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Finitary Lie algebras

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Abstract

An algebra is called finitary if it consists of finite-rank transformations of a vector space. We classify finitary simple and finitary irreducible Lie algebras over an algebraically closed field of characteristic $\neq 2, 3$.

1. Introduction

Let V be a vector space over a field $\mathbb{F}$. An element $x \in \operatorname{End}_{\mathbb{F}} V$ is called *finitary* if $\dim xV < \infty$. The finitary transformations of V form an ideal $\mathfrak{fgl}(V)$ of the Lie algebra $\mathfrak{gl}(V)$ of all linear transformations of V. A Lie algebra is called *finitary* if it is isomorphic to a subalgebra of $\mathfrak{fgl}(V)$ for some V. The aim of this paper is to classify infinite-dimensional finitary simple and finitary irreducible Lie algebras over an algebraically closed field of characteristic different from 2 and 3. This extends the results of the first author [2–5] solving the problem over an arbitrary field of characteristic 0. However the present approach is completely different and use many beautiful ideas from [10,11] and [14–17]. This also gives an easier

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and very direct access to characteristic zero case. We prove the following main theorem.

Theorem 1.1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $\neq 2, 3$. Then any infinite-dimensional finitary simple Lie algebra over $\mathbb{F}$ is isomorphic to one of the following*:

(1) *a finitary special linear algebra* $\mathfrak{fsl}(V, \Pi)$;
(2) *a finitary orthogonal algebra* $\mathfrak{fo}(V, \Phi)$;
(3) *a finitary symplectic algebra* $\mathfrak{fsp}(V, \Psi)$.

Here V is a vector space over $\mathbb{F}$; Φ (respectively Ψ) is a non-degenerate symmetric (respectively skew-symmetric) form on V; and Π is a total subspace of the dual V^*.

The definition of the algebras (1)–(3) is given in Section 4. These algebras are simple and can be represented as the direct limits of natural embeddings of the corresponding classical finite-dimensional Lie algebras. Note that Theorem 1.1 is similar to Hall's classification [10] of simple locally finite groups of finitary linear transformations. If we restrict ourselves to the case of countably dimensional algebras, we get more precise result.

Corollary 1.2. *Any finitary simple Lie algebra of (infinite) countable dimension over an algebraically closed field of characteristic $\neq 2, 3$ is isomorphic to one of the following three algebras*: $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, *and* $\mathfrak{sp}(\infty)$.

Recall that $\mathfrak{sl}(\infty)$ is the union of the chain of natural embeddings

$$\mathfrak{sl}(2) \to \mathfrak{sl}(3) \to \cdots \to \mathfrak{sl}(n) \to \cdots$$

and $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ are constructed similarly. These algebras often appear in the literature in various context. One of the recent relevant results is the classification of locally finite split simple Lie algebras obtained by Neeb and Stumme [19]. They proved that for each infinite cardinality J there exist at most three isomorphy classes of such algebras over a given field of characteristic zero: $\mathfrak{sl}(J)$, $\mathfrak{o}(J)$, and $\mathfrak{sp}(J)$. These algebras are finitary and isomorphic to $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$, respectively, for countable J.

We also classify finitary irreducible Lie algebras.

Theorem 1.3. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $\neq 2, 3$, V an infinite-dimensional vector space over $\mathbb{F}$, and $L \subset \mathfrak{fgl}(V)$ a finitary irreducible Lie subalgebra. Then there exists a non-degenerate symmetric or skew-symmetric form on V or a total subspace Π of the dual V^* such that $L = \mathfrak{fgl}(V, \Pi)$, $\mathfrak{fsl}(V, \Pi)$, $\mathfrak{fo}(V, \Phi)$, or $\mathfrak{fsp}(V, \Psi)$. In particular, $[L, L]$ is simple.*

The paper is organized as follows. In Section 2 we develop a technique of small rank generators. Among other things we prove that for a finite-dimensional vector space V each proper irreducible subalgebra of $\mathfrak{gl}(V)$ generated by transformations of relatively small rank with respect to $\dim V$ is either $\mathfrak{sl}(V)$, or $\mathfrak{o}(V,\Phi)$ or $\mathfrak{sp}(V,\Psi)$ for some non-degenerate forms Φ and Ψ. One of the main difficulties here is to eliminate Cartan type Lie algebras. Similar result for groups has been proved by Hall, Liebeck, and Seitz [11, Theorem 4].

It is easy to show that each finitary Lie algebra L is locally finite, i.e. L is the direct limit of a family of its finite-dimensional subalgebras L_α. Such a family $\{L_\alpha\}$ is called a local system of L. Our aim in Section 3 is to construct for irreducible L a nicely behaving local system $\{L_\beta\}$ such that each L_β is "close" to a classical simple Lie algebra. We combine the technique developed in Section 2 with ideas of Leinen and Puglisi [14–17]. In Section 4 we prove the main theorems, "recovering" L from the local data $\{L_\beta\}$.

1.1. Notation

Throughout the paper $\mathbb{F}$ is the ground field. For a finite-dimensional Lie algebra L we denote by $L^{(\infty)}$ the smallest member of the derived series of L. Note that $L^{(\infty)}$ is *perfect*, i.e. $[L,L]=L$. By $\operatorname{rad} L$ we denote the solvable radical of L; $C(L)$ is the centre of L. If V is a fixed L-module, we denote by $\operatorname{nil} L$ the nil radical of the corresponding representation, i.e. $\operatorname{nil} L$ is the maximal ideal of L such that each element from $\operatorname{nil} L$ acts nilpotently on V. If V is a vector space and $x \in \operatorname{End} V$ (or $x \in L$ and V is an L-module), we denote by $\operatorname{rk} x$ the dimension of xV.

2. Small rank generators

In this section we develop a technique of small rank generators adapting some ideas from [10,11] for the Lie algebra case. Let in this section W be any vector space over an arbitrary field $\mathbb{F}$, G a subalgebra of $\mathfrak{gl}(W)$ and $h \in \operatorname{End} W$. The transformation h is called *locally finite* if every finite subset of W is contained in a h-invariant finite-dimensional subspace. Clearly, every finitary transformation is locally finite, and if G is locally finite and $x \in G$, then $\operatorname{ad} x$ is a locally finite transformation of $\operatorname{End} G$. The *Fitting subspaces* of W with respect to h are defined as

$$W^0(h) := \bigcup_{n\geqslant 1} \ker h^n, \qquad W^1(h) := \bigcap_{n\geqslant 1} h^n W.$$

Lemma 2.1. (1) *If $h \in \operatorname{End} W$ is locally finite, then W decomposes*

$$W = W^0(h) \oplus W^1(h).$$

(2) *Let h act locally finitely on vector spaces W_1, W_2, W_3 and let $\varphi : W_1 \otimes W_2 \to W_3$ be a h-invariant homomorphism of vector spaces* (*i.e.* $\varphi(hv_1 \otimes v_2 + v_1 \otimes hv_2) = h\varphi(v_1 \otimes v_2)$). *Then*

$$\varphi\bigl(W_1^0(h) \otimes W_2^i(h)\bigr) \subset W_3^i(h) \quad (i = 0, 1).$$

(3) *Suppose G is locally finite and $h \in G$. Then $G^1(\operatorname{ad} h) + [G^1(\operatorname{ad} h), G^1(\operatorname{ad} h)]$ is an ideal of G. If $\operatorname{ad} h$ is not locally nilpotent, then this ideal is not central in G.*

(4) *Suppose G is locally finite and $h \in G$ is locally finite. Then*

$$G^0(h)W^i(h) \subset W^i(h) \quad (i = 0, 1).$$

Proof. Let $\{W_\alpha \mid \alpha \in A\}$ be the set of all h-invariant finite-dimensional subspaces of W. For each W_α we have the Fitting decomposition $W_\alpha = W_\alpha^0(h) \oplus W_\alpha^1(h)$. Clearly,

$$W_\alpha^i(h) \subset W_\beta^i(h) \quad (i = 0, 1)$$

whenever $W_\alpha \subset W_\beta$. Hence $W^i(h) = \bigcup_{\alpha \in A} W_\alpha^i(h)$ $(i = 0, 1)$, and the claim follows.

(2) The claim can be checked in finite-dimensional h-invariant subspaces where this is known.

(3), (4) follow from (2) setting $W_1 = G$, $W_2 = W_3 = G, W$. □

Lemma 2.2. (1) *Let $x, y \in \mathfrak{gl}(W)$. Then* $\operatorname{rk}[x, y] \leqslant 2 \operatorname{rk} x$.

(2) *Let G be generated by a finite set $\{x_1, \ldots, x_n\}$ of elements of finite rank. Then*

$$\dim GW \leqslant \sum_{i=1}^{n} \operatorname{rk} x_i.$$

Proof. (1) We have

$$\operatorname{rk}[x, y] = \dim[x, y]W \leqslant \dim xyW + \dim yxW.$$

Clearly, $\dim xyW \leqslant \dim xW = \operatorname{rk} x$. Considering the surjective linear mapping $y : xW \to y(xW)$ we obtain $\dim yxW \leqslant \dim xW = \operatorname{rk} x$.

(2) Obviously $GW = \sum_{i=1}^n x_i W$. Thus $\dim GW \leqslant \sum_{i=1}^n \operatorname{rk} x_i$. □

Proposition 2.3 (see [20]). (1) *Let G be a finitely generated subalgebra of $\mathfrak{fgl}(W)$. Then* $\dim GW < \infty$ *and* $\dim(W/\operatorname{ann}_W G) < \infty$.

(2) *$\mathfrak{fgl}(W)$ is locally finite.*

Proof. (1) Let $\{x_1, \ldots, x_n\}$ generate G. Since all x_i have finite rank, by Lemma 2.2(2), GW is finite-dimensional. We have $\operatorname{ann}_W G = \bigcap_{i=1}^n \operatorname{ann}_W x_i$. Therefore

$$\dim(W/\operatorname{ann}_W G) \leqslant \sum_{i=1}^n \dim(W/\operatorname{ann}_W x_i) = \sum_{i=1}^n \dim x_i W = \sum_{i=1}^n \operatorname{rk} x_i < \infty.$$

(2) Let G be as above. Clearly, the homomorphism

$$G \to \operatorname{Hom}(W/\operatorname{ann}_W G, GW), \quad g \mapsto (v + \operatorname{ann}_W G \mapsto gv)$$

is injective. This implies $\dim G \leqslant \dim(W/\operatorname{ann}_W G) \cdot \dim(GW) < \infty$. Therefore G is finite-dimensional and $\mathfrak{fgl}(W)$ is locally finite. $\square$

Remark 2.4. For any Lie subalgebra G of $\mathfrak{gl}(W)$ let $\operatorname{nil} G$ denote the sum of all ideals of G consisting of nilpotent endomorphisms. Let G be an irreducible subalgebra of $\mathfrak{gl}(W)$ and $\dim W < \infty$. Then $\operatorname{nil} G = (0)$. Namely, by Engel's theorem $U := \operatorname{ann}_W(\operatorname{nil} G) \neq (0)$. As U is G-invariant and W is G-irreducible, the result follows.

Next we investigate composition series. The following simple observation seems first have been made by Block [8, Lemma 2.1], see also [17, Proposition 1].

Lemma 2.5. *Let G be an irreducible subalgebra of $\mathfrak{gl}(W)$ and S an ideal of G. If W has a finite S-composition series, then all factors are S-isomorphic.*

Lemma 2.6. *Let G be an irreducible subalgebra of $\mathfrak{gl}(W)$, S a non-zero ideal of G, and H a non-nil subalgebra of G. Assume* $\dim W < \infty$. *Then S is not nil. Let X be a generating set for H,*

$$d^H := \max\{\operatorname{rk} x \mid x \in X\},$$
$$d_S := \min\{\operatorname{rk} u \mid u \in S \setminus \operatorname{nil} S\}.$$

Then either $\dim W \leqslant d^H d_S$ *or every S-submodule of W is H-invariant.*

Proof. Let

$$W = W_1 \supset \cdots \supset W_t \supset (0)$$

be a S-composition series of W. By Lemma 2.5, all composition factors are isomorphic to W_t.

(a) Remark 2.4 implies that S is not nil. Since $W_i/W_{i+1} \cong W_t$ for all i, it is easy to see that $\operatorname{ann}_S W_t \subset \operatorname{nil} S$. Take $u \in S \setminus \operatorname{nil} S$ arbitrary. Then $uW_t \neq (0)$ and hence $u(W_i/W_{i+1}) \neq (0)$ for all i. Therefore $uW_i \not\subset W_{i+1}$. This proves that $t \leqslant d_S$.

(b) The Jordan–Hölder Theorem implies that every factor of every S-composition series of W is isomorphic to W_t. Suppose there is a S-submodule

U of W which is not H-invariant. Then there is $x \in X$ such that $xU \not\subset U$. Since $xU + U$ is a S-module and

$$\dim(xU + U/U) \leqslant \dim xU \leqslant \operatorname{rk} x \leqslant d^H,$$

this shows that $\dim W_t \leqslant d^H$. Consequently,

$$\dim W \leqslant t \cdot \dim W_t \leqslant d_S d^H. \quad \square$$

Proposition 2.7. *Let G be an irreducible subalgebra of $\mathfrak{gl}(W)$ generated by elements of rank $\leqslant d$.*

(1) *Suppose* $4d < \dim W < \infty$. *Then* $\operatorname{rad} G = C(G)$.
(2) *Suppose* $8d^2 < \dim W < \infty$. *Then G has a unique minimal non-central ideal S. Moreover, $S = S^{(1)}$, $S/C(S)$ is simple, S acts irreducibly on W and S is spanned by elements of rank $\leqslant 2d$.*

Proof. (1) Suppose $\operatorname{char} \mathbb{F} = 0$. Then by Lie's theorem $[\operatorname{rad} G, G] \subset \operatorname{nil} G = (0)$. Hence $\operatorname{rad} G = C(G)$.

Suppose $\operatorname{char} \mathbb{F} = p > 0$. Assume $\operatorname{rad} G \neq C(G)$. Then there is an ideal I of G satisfying

$$C(G) \subsetneq I, \qquad I^{(1)} \subset C(G).$$

Let X be a generating set of G, so that all elements of X have rank $\leqslant d$. Then $J := [I, X] \subset I$ is an ideal of G. According to Lemma 2.2, J is spanned by elements of rank $\leqslant 2d$.

By Schur's lemma every non-zero element of $C(G)$ acts invertibly on W. Hence if $a \in C(G)$, $a \neq 0$, then $\operatorname{rk} a = \dim W > 4d$.

Take $a, b \in J$ of rank $\leqslant 2d$. Then $[a, b] \in I^{(1)} \subset C(G)$ and $\operatorname{rk}[a, b] \leqslant 4d$. This shows that $[a, b] = 0$. Therefore, $J^{(1)} = (0)$. Let $a \in J$ be an element of rank $\leqslant 2d$. Then $[a^p, G] \subset J^{(1)} = (0)$, thus $a^p \in C(G)$. As before, this implies $a^p = 0$. As a consequence, J is an abelian ideal spanned by nil elements. But then J is nil, i.e. $J \subset \operatorname{nil} G = (0)$. This in turn means $[I, G] = J = (0)$, a contradiction.

(2) Let X be a set of generators of G of rank $\leqslant d$. By (1) we have $\operatorname{rad} G = C(G)$. Observe that $G \neq C(G)$ (otherwise by Schur's lemma every element of G acts invertibly on W, i.e. has rank $\dim W > d$). Let S be a minimal non-central ideal of G. Then $S^{(1)} = S$. We have

$$S = S^{(1)} \subset [G, S] = [X, S] \subset S,$$

so $S = [X, S]$. In particular, S is spanned by elements of rank $\leqslant 2d$. Set in Lemma 2.6 $H := G$. Note that $d_S \leqslant 2d$. As $\dim W > 8d^2 > d_S d$, the theorem implies that W is S-irreducible. Since S is spanned by elements of rank $\leqslant 2d$ and $\dim W > 8d$, the first part of this lemma yields that $\operatorname{rad} S = C(S)$. By definition of S, the image of S in $G/\operatorname{rad} G$ is a minimal ideal. Block's result [8] on minimal ideals in semi-simple algebras proves that $S/\operatorname{rad} S$ is simple. Let J be any other minimal non-central ideal of G. As above, J is spanned by elements of rank $\leqslant 2d$. We have $[S, J] \subset C(G)$, hence $[S, J] = [S, J^{(1)}] = (0)$. Since S is irreducible, by Schur's lemma, all elements of J act invertibly on W, i.e. have rank $> 2d$. The contradiction obtained shows that S is unique. $\square$

If in the setting of Proposition 2.7 the characteristic of $\mathbb{F}$ is 0, then we obtain that S is simple and $G = S \oplus C(G)$. In positive characteristic we apply the BLOCK–WILSON–STRADE–PREMET classification of simple finite-dimensional Lie algebras to obtain an analogous result.

Theorem 2.8 [21]. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 3$ and G a finite-dimensional simple Lie algebra over $\mathbb{F}$. Then G is of classical, Cartan, or Melikian type.*

We will not describe the mentioned algebras in detail. Let $G_{\mathbb{C}}$ denote a finite-dimensional simple Lie algebra over $\mathbb{C}$. Using a Chevalley basis and reducing modulo p gives a Lie algebra over $GF(p)$ and tensoring with $\mathbb{F}$ gives a Lie algebra over $\mathbb{F}$ of type

$$A_n,\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ \text{or } G_2.$$

These algebras are simple (if $p > 3$) except in case A_n with $p|n+1$. In this case $\mathfrak{psl}(n+1) = \mathfrak{sl}(n+1)/\mathbb{F}E_{n+1}$ is simple. These simple Lie algebras are called "classical." The Cartan type Lie algebras are described in [24] or [22]. There are four series

$$W(m;\underline{n}),\quad S(m;\underline{n};\Phi)^{(1)}\ (m \geqslant 3),\quad H(m;\underline{n};\Phi)^{(2)}\ (m = 2r),$$
$$K(m;\underline{n};\Phi)^{(1)}\ (m = 2r+1)$$

named Witt, Special, Hamiltonian, Contact. In general these are difficult to handle. Fortunately, for our purposes the graded algebras in these classes are of interest only. Namely, every Cartan type Lie algebra L carries a natural filtration and they fulfill the "compatibility condition"

$$X(m;\underline{n})^{(2)} \subset \operatorname{gr} L \subset X(m;\underline{n})$$

where X is one of W, S, H, K. The graded algebras $W(m;\underline{n})$, $S(m;\underline{n})$, $H(m;\underline{n})$, $K(m;\underline{n})$ are described in more detail in [23]. We will eventually refer to that description. The Melikian algebras only occur in characteristic 5. They carry a natural grading

$$\mathcal{G}(n_1, n_2) = \bigoplus_{i=-3}^{3(5^{n_1}+5^{n_2}-1)} \mathcal{G}(n_1, n_2)_i.$$

A description of this can be found in [22].

Let G be any finite-dimensional simple Lie algebra, $G_{(0)}$ a maximal subalgebra and $G_{(-1)} \supset G_{(0)}$ a subspace minimal subject to the condition $[G_{(0)}, G_{(-1)}] \subset G_{(-1)}$. Such a pair $(G_{(0)}, G_{(-1)})$ defines a *standard filtration* by

$$G_{(i)} := \left\{x \in G_{(i-1)} \,\middle|\, [G_{(-1)}, x] \subset G_{(i-1)}\right\} \quad (i > 0),$$
$$G_{(i)} := [G_{(i+1)}, G_{(-1)}] + G_{(i+1)} \quad (i < -1).$$

Since G is finite-dimensional and $G_{(0)}$ is a maximal subalgebra, there is $r > 0$ such that $G = G_{(-r)}$. Since G is simple, there is $s \geqslant 0$ with $G_{(s+1)} = (0)$.

Theorem 2.9. *Let $G \subset \mathfrak{gl}(W)$ be an irreducible Lie algebra such that*

(1) $\overline{G} := G/C(G)$ *is simple*;
(2) $\overline{G}$ *has a standard filtration*

$$\overline{G} = \overline{G}_{(-r)} \supset \cdots \supset \overline{G}_{(s)}, \quad r, s \geqslant 1;$$

(3) *there is a set $X \subset G$ of elements of rank $\leqslant d$ for which*

$$\overline{\operatorname{span} X} + \overline{G}_{(0)} = \overline{G}_{(-1)};$$

(4) $\overline{G}_{(0)} = [\overline{G}_{(-1)}, \overline{G}_{(1)}] + \overline{G}_{(1)}$.

Then

$$\dim W \leqslant d \dim \overline{G}_{(-1)}/\overline{G}_{(0)} + 2d \max\left\{\dim \overline{G}_{(-1)}/\overline{G}_{(0)},\ 2\dim \overline{G}_{(s)}\right\}.$$

Proof. Let $G_{(i)}$ be the full preimage of $\overline{G}_{(i)}$ $(-r \leqslant i \leqslant s)$. Then

$$G = G_{(-r)} \supset \cdots \supset G_{(s+1)} = C(G)$$

and $G_{(-1)} = \operatorname{span} X + G_{(0)}$.

(a) If $[G_{(s)}, [G_{(1)}, X]] \not\subset C(G)$, then the space

$$U := \sum_{n \geqslant 0} (\operatorname{ad} X)^n \left(\left[G_{(s)}, [G_{(1)}, X]\right]\right) + C(G)$$

is X-invariant by construction, and $G_{(0)}$-invariant since $[G_{(0)}, X] \subset \operatorname{span} X + G_{(0)}$. As $\operatorname{span} X + G_{(0)}$ generates G, U is an ideal of G. It contains $C(G)$ properly. As $G/C(G)$ is simple, we obtain $U = G$. Choose $y_1, \ldots, y_t \in [G_{(s)}, [G_{(1)}, X]]$ which have rank $\leqslant 4d$ (see Lemma 2.2) and for which $\{\bar{y}_1, \ldots, \bar{y}_t\}$ spans $\overline{[G_{(s)}, [G_{(1)}, X]]}$. Observe that $\overline{[G_{(s)}, [G_{(1)}, X]]} \subset \overline{G}_{(s)}$. Thus we may assume $t \leqslant \dim \overline{G}_{(s)}$. Let G' be the algebra generated by $X \cup \{y_1, \ldots, y_t\}$. Observe that $G' + C(G) = G$. Therefore G' is an ideal of G, so $G'W$ is a non-zero G-submodule of W. Thus $G'W = W$. Lemma 2.2 now implies

$$\dim W \leqslant d(\dim \operatorname{span} X + 4t) \leqslant d\left(\dim \overline{G}_{(-1)}/\overline{G}_{(0)} + 4\dim \overline{G}_{(s)}\right).$$

(b) Suppose $[G_{(s)}, [G_{(1)}, X]] \subset C(G)$. By (4), $[G_{(s)}, G_{(0)}] \subset C(G)$. Take $u \in G_{(s)} \backslash C(G)$ and let

$$U := \sum_{n \geqslant 0} (\operatorname{ad} X)^n \bigl([u, X]\bigr) + C(G).$$

The present assumption implies that U is stable under $G_{(0)}$. Hence it is an ideal of G containing $C(G)$. Consequently, $U = G$ or $U = C(G)$. Suppose $U = G$. Then we proceed as before and obtain the estimate

$$\dim W \leqslant d\bigl(\dim \overline{G}_{(-1)}/\overline{G}_{(0)} + 2 \dim \overline{G}_{(-1)}/\overline{G}_{(0)}\bigr).$$

Suppose $U = C(G)$. Then $[u, X] \subset C(G)$. Therefore the set

$$J := \bigl\{g \in G_{(s)} \bigm| [g, X] \subset C(G)\bigr\}$$

is a subspace of $G_{(s)}$ which contains $C(G)$ properly. As $[G_{(0)}, G_{(s)}] \subset C(G)$, J is invariant under $G_{(0)}$ and X, hence is an ideal of G. This implies $J = G$, so this final case is impossible. □

The canonical filtration of the Cartan type Lie algebras is a standard filtration. The filtration defined by the depth 3 gradation of the Melikian algebra is a standard filtration as well. In both cases the property (4) of Theorem 2.9 holds. Thus in order to apply Theorem 2.9 it suffices to determine $\dim \operatorname{gr}_{-1} G$ and $\dim \operatorname{gr}_s G$.

Corollary 2.10. *Assume* $\operatorname{char} \mathbb{F} = p > 2$. *Let* $G \subset \mathfrak{gl}(W)$ *be an irreducible Lie algebra such that*

(1) $\overline{G} := G/C(G)$ *is simple of Cartan or Melikian type*;
(2) *there is a set* $X \subset G$ *of elements of rank* $\leqslant d$ *for which*

$$\overline{\operatorname{span} X} + \overline{G}_{(0)} = \overline{G}_{(-1)}$$

in the canonical (*or depth* 3) *filtration.*

Then $\dim W$ *is bounded by*

$$\begin{array}{ll} 5dm & \textit{if } \overline{G} = W(m; \underline{n}); \\ dm(2m-1) & \textit{if } \overline{G} = S(m; \underline{n}; \Phi)^{(1)},\ m \geqslant 3; \\ 5dm & \textit{if } \overline{G} = H(m; \underline{n}; \Phi)^{(2)},\ m = 2r \geqslant 2; \\ 5d(m-1) & \textit{if } \overline{G} = K(m; \underline{n}; \Phi)^{(1)},\ m = 2r+1 \geqslant 3; \\ 10d & \textit{if } \overline{G} \textit{ is Melikian.} \end{array}$$

Proof. For $a = (a_1, \ldots, a_m)$ set $|a| := \sum a_i$. Let $\tau = \tau(\underline{n}) := (p^{n_1} - 1, \ldots, p^{n_m} - 1)$.

(a) Suppose $\overline{G} = W(m; \underline{n})$. Then $\overline{G}$ is graded,

$$\overline{G}_{-1} = \sum_{i=1}^{m} \mathbb{F} D_i, \qquad \overline{G}_s = \sum_{i=1}^{m} \mathbb{F} x^{(\tau)} D_i$$

(see [23, (4.2.2)]). Thus $\dim \overline{G}_{-1} = \dim \overline{G}_s = m$.

(b) Suppose $\overline{G} = S(m; \underline{n}; \Phi)^{(1)}$. Then

$$S(m; \underline{n})^{(1)} \subset \operatorname{gr} \overline{G} \subset S(m; \underline{n}) = S(m; \underline{n})^{(1)} + \sum_{j=1}^{m} \mathbb{F} x^{(\tau-(p^{n_j}-1)\varepsilon_j)} D_j$$

(see the proof of [23, (4.3.7)]). Also

$$S(m; \underline{n})^{(1)}{}_{(s)} = \sum_{i<j} \mathbb{F}\bigl(x^{(\tau-\varepsilon_i)} D_j - x^{(\tau-\varepsilon_j)} D_i\bigr)$$

(see [23, (4.3.3)]). This implies that $s = |\tau| - 2 > |\tau - (p^{n_j} - 1)\varepsilon_j| - 1$ for all j (as $p > 2$), hence $\operatorname{gr}_s \overline{G} = S(m; \underline{n})^{(1)}{}_{(s)}$. Thus $\dim \operatorname{gr}_{-1} \overline{G} = m$, $\dim \operatorname{gr}_s \overline{G} = m(m-1)/2$.

(c) Suppose $\overline{G} = H(m; \underline{n}; \Phi)^{(2)}$, $m = 2r \geqslant 2$. Then

$$\begin{aligned} H(m; \underline{n})^{(2)} &\subset \operatorname{gr} \overline{G} \subset H(m; \underline{n}) \\ &= H(m; \underline{n})^{(2)} + \mathbb{F} D_H\bigl(x^{(\tau)}\bigr) + \sum_{i=1}^{m} \mathbb{F} D_H\bigl(x^{(p^{n_i})}\bigr). \end{aligned}$$

Since $H(m; \underline{n})^{(2)}{}_{(|\tau|-3)} \neq 0$ (see [23, (4.4.4)]) and, for all i, $|\tau| - 3 > p^{n_i} - 2 = \deg D_H(x^{(p^{n_i})})$, it is only possible that

$$\begin{aligned} \operatorname{gr}_s \overline{G} &= \mathbb{F} D_H\bigl(x^{(\tau)}\bigr) \quad \text{or} \\ \operatorname{gr}_s \overline{G} &= H(m; \underline{n})^{(2)}{}_{|\tau|-3} = \sum_{i=1}^{m} \mathbb{F} D_H\bigl(x^{(\tau-\varepsilon_i)}\bigr). \end{aligned}$$

Thus $\dim \operatorname{gr}_s \overline{G} \in \{1, m\}$ and $\dim \operatorname{gr}_{-1} \overline{G} = m$.

(d) Suppose $\overline{G} = K(m; \underline{n}; \Phi)^{(1)}$, $m = 2r + 1 \geqslant 3$. Then

$$K(m; \underline{n})^{(1)} \subset \operatorname{gr} \overline{G} \subset K(m; \underline{n}) = K(m; \underline{n})^{(1)} + \mathbb{F} D_K\bigl(x^{(\tau)}\bigr)$$

(see [23, (4.5.4)]). Thus it is only possible that

$$\begin{aligned} \operatorname{gr}_s \overline{G} &= \mathbb{F} D_K\bigl(x^{(\tau)}\bigr) \quad \text{or} \\ \operatorname{gr}_s \overline{G} &= K(m; \underline{n})^{(1)}{}_s = \sum_{i=1}^{2r} \mathbb{F} D_K\bigl(x^{(\tau-\varepsilon_i)}\bigr). \end{aligned}$$

Thus $\dim \operatorname{gr}_s \overline{G} \in \{1, m-1\}$. Note that $K(m; \underline{n})_{-2}$ is 1-dimensional, and $\dim \operatorname{gr}_{-1} \overline{G} = \dim K(m; \underline{n})_{-1} = m - 1$.

(e) Suppose $\overline{G}$ is a Melikian algebra. The depth 3 grading has the property that $\dim \overline{G}_{-1} = 2$, $\dim \overline{G}_s = 2$ (see [22, §3.6]). □

Corollary 2.11. *Assume* $\operatorname{char} \mathbb{F} = p > 3$. *Let* $G \subset \mathfrak{gl}(W)$ *be an irreducible Lie algebra such that*

(1) $\overline{G} = G/C(G)$ *is simple of Cartan or Melikian type*;
(2) G *is spanned by elements of rank* $\leqslant d$ $(d \geqslant 2)$.

Then

$$\dim W \leqslant \min\bigl\{5(28/\ln p)^2 d(\ln d)^2, 2^{11}d^2\bigr\} \quad \textit{if } \overline{G} \textit{ is of Cartan type};$$
$$\dim W \leqslant 10d \quad \textit{if } \overline{G} \textit{ is of Melikian type}.$$

Proof. In view of Corollary 2.10 it suffices to consider only Cartan type Lie algebras. We shall use Corollary 2.10, the trivial facts that $\dim \overline{G} \leqslant (\dim W)^2$ and $\dim G = \dim \operatorname{gr} G$, and the compatibility condition

$$X(m;\underline{1})^{(2)} \subset X(m;\underline{n})^{(2)} \subset \operatorname{gr} \overline{G}, \quad X = W, S, H, K,$$

in combination with the well known dimensions of $X(m;\underline{1})^{(2)}$, namely (see [23, 4.3.7, 4.4.5]:

$$\dim X(m;\underline{1})^{(2)} = \begin{cases} mp^m & \text{if } X = W; \\ (m-1)(p^m-1) & \text{if } X = S; \\ p^m - 1 - \delta_{m,2} & \text{if } X = H; \\ p^m - \delta,\ \delta \in \{0,1\} & \text{if } X = K. \end{cases}$$

(a) Suppose $\overline{G} = W(m;\underline{n})$. Then $\overline{G} = \overline{G}_{(-1)}$ and Corollary 2.10 shows that $\dim W \leqslant 5dm$. Therefore

$$mp^m \leqslant \dim \overline{G} \leqslant (5dm)^2.$$

(b) Suppose $\overline{G} = S(m;\underline{n};\Phi)^{(1)}$. Then $\overline{G} = \overline{G}_{(-1)}$ and Corollary 2.10 shows that $\dim W \leqslant dm(2m-1)$. Therefore

$$(m-1)(p^m-1) \leqslant \dim \overline{G} \leqslant \bigl(dm(2m-1)\bigr)^2.$$

As a weak estimate it follows that $p^m \leqslant 4d^2m^3$.

(c) Suppose $\overline{G} = H(m;\underline{n};\Phi)^{(2)}$. Then $\overline{G} = \overline{G}_{(-1)}$ and Corollary 2.10 shows that $\dim W \leqslant 5dm$. Therefore

$$(p^m - 2) \leqslant \dim \overline{G} \leqslant 25d^2m^2.$$

(d) Suppose $\overline{G} = K(m;\underline{n};\Phi)^{(1)}$. Then $\overline{G} = \overline{G}_{(-2)}$ and $\dim \overline{G}/\overline{G}_{(-1)} = 1$. Let B be a basis of G consisting of elements of rank $\leqslant d$. Choose $b_1 \in B$ such that $\overline{G} = \overline{G}_{(-1)} \oplus \mathbb{F}\bar{b}_1$. Then for every $b \in B$ there is $\alpha(b) \in \mathbb{F}$ such that

$\bar{b} - \alpha(b)\bar{b}_1 \in \overline{G}_{(-1)}$. Since $\operatorname{rk}(\bar{b} - \alpha(b)\bar{b}_1) \leqslant 2d$, Corollary 2.10 shows that $\dim W \leqslant 10d(m-1)$. Therefore

$$(p^m - 1) \leqslant \dim \overline{G} \leqslant 10^2 d^2 (m-1)^2.$$

As $m \geqslant 3$ in this case one has $10^2 d^2 (m-1)^2 < 5^2 d^2 m^3$.

(e) As a weak estimate we obtain in all cases

$$\dim W \leqslant 5dm^2, \qquad p^m \leqslant 5^2 d^2 m^3. \tag{1}$$

Note that $5^3 > 3^4$. Hence

$$m^3 \leqslant 3^m < 5^{(3/4)m} \leqslant p^{m(1-1/4)} \quad \text{for } m \in \mathbb{N}.$$

As $d \geqslant 2$, this gives

$$p^{m/4} \leqslant p^m/m^3 \leqslant 5^2 d^2 \leqslant d^7.$$

Hence $m \leqslant 28(\ln p)^{-1} \ln d$, and (1) then yields

$$\dim W \leqslant 5\left(\frac{28}{\ln p}\right)^2 d(\ln d)^2.$$

In order to obtain the final estimate observe that

$$p^{5\sqrt{d}} > \mathrm{e}^{5\sqrt{d}} > \frac{1}{4!}\left(5\sqrt{d}\right)^4 > 25d^2 \geqslant p^{m/4},$$

so $m^2 < 20^2 d$. Therefore by (1),

$$\dim W \leqslant 5dm^2 < 2000d^2 < 2^{11} d^2,$$

as required. □

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$. We say that a finite-dimensional Lie algebra L over $\mathbb{F}$ is a *Chevalley algebra* if L is obtained from a simple complex Lie algebra by "reduction modulo p" (see [9]), i.e. L has a Chevalley basis $\{x_\alpha, h_\beta \mid \alpha \in R, \beta \in B\}$ with all structure constants in $\mathbb{Z}_p$ (here R is the root system of L and B is a base of R). For $p > 3$ the algebra L is simple except in the case where L is of type A_r and $p|(r+1)$. In the exceptional case $L/C(L)$ is simple. The algebra L has a natural p-mapping: $x_\alpha^{[p]} = 0$ and $h_\beta^{[p]} = h_\beta$ for all $\alpha \in R$ and all $\beta \in B$, respectively. By *rank* of L we mean the rank of the corresponding root system R.

Theorem 2.12. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $\neq 2, 3$ and let L be a finite-dimensional Lie algebra over $\mathbb{F}$. Assume that L is simple if* $\operatorname{char}\mathbb{F} = 0$, *or L is a Chevalley algebra if* $\operatorname{char}\mathbb{F} = p > 3$. *Let r be the rank of L and let V be an irreducible finite-dimensional L-module. Set $d = \min\{\operatorname{rk} x \mid 0 \neq x \in L\}$. Then the following holds.*

(1) $\dim V \leqslant 2^7 dr$.
(2) *If* $\dim V > 2^7 d$, *then the* L*-module* V *is restricted* (*for* $\operatorname{char}\mathbb{F} > 3$).
(3) *If* $\dim V > 2^{15} d^2$, *then* $L = \mathfrak{sl}(V)$, $\mathfrak{o}(V, \Phi)$, *or* $\mathfrak{sp}(V, \Phi)$ *where* Φ *is a non-degenerate symmetric* (*respectively non-symmetric*) *bilinear form on* V.

Proof. Let $L = H \bigoplus_{\alpha\in R} L_\alpha$ be the root decomposition of L and let the $x_\alpha \in L_\alpha$ be the standard Chevalley generators of L. First we are going to find a non-zero $h \in H$ with $\operatorname{rk} h \leqslant 2^4 d$. Fix any $x \in L$ with $\operatorname{rk} x = d$. We have $x = h(x) + \sum_{\alpha\in R} k_\alpha(x)x_\alpha$ where $h(x) \in H$ and $k_\alpha(x) \in \mathbb{F}$. If all $k_\alpha(x) = 0$, then we can take $h = x$. Therefore we can assume that there exists $\beta \in R$ with $k_\beta(x) \neq 0$. If the root β is long, then set $\mu = \beta$ and $y = x$. Otherwise there exists a short root γ such that $\beta + \gamma$ is a long root and we set $\mu = \beta + \gamma$ and $y = [x, x_\gamma]$. In both cases we have $\operatorname{rk} y \leqslant 2d$ and $k_\mu(y) \neq 0$ for a long root μ. Since μ is long, for any root $\alpha \neq \mu$ we have $|2\mu - \alpha| > |\mu|$, so $\alpha - 2\mu$ is not a root. Therefore

$$\big[x_\mu, \big[x_{-\mu}, [x_{-\mu}, y]\big]\big] = [x_\mu, 2k_\mu(y)x_{-\mu}] = 2k_\mu(y)h_\mu.$$

Set $h = h_\mu$. By Lemma 2.2(1), $\operatorname{rk} h \leqslant 2^3 \operatorname{rk} y \leqslant 2^4 d$, as required.

Set $R_1 = \{\alpha \in R \mid \alpha(h) \neq 0\}$ and $L_1 = \langle x_\alpha \mid \alpha \in R_1\rangle_{\mathbb{F}}$. Clearly $\operatorname{rk} x_\alpha \leqslant 2^5 d$ for all $\alpha \in R_1$. Since $L/C(L)$ is simple, by Lemma 2.1(3), we can represent L in the form $L = L_1 + [L_1, L_1]$. Therefore for any $\alpha \in R$ we have $x_\alpha = k[x_\beta, x_\gamma]$ for some $k \in \mathbb{F}$ and $\beta, \gamma \in R_1$, so $\operatorname{rk} x_\alpha \leqslant 2^6 d$. Since $h_\alpha = [x_\alpha, x_{-\alpha}]$, we have $\operatorname{rk} h_\alpha \leqslant 2^7 d$ for all $\alpha \in R$. Let $\{\alpha_1, \dots, \alpha_r\}$ be a base of the root system R. Since the elements $x_{-\alpha_1}, \dots, x_{-\alpha_r}, x_{\alpha_1}, \dots, x_{\alpha_r}$ generate L, by Lemma 2.2(2), $\dim V \leqslant 2^6 d \cdot 2r = 2^7 dr$, so (1) holds.

(2) Let $\operatorname{char}\mathbb{F} = p > 3$. Recall that $x_\alpha^{[p]} = 0$ and $h_\alpha^{[p]} = h_\alpha$ for all $\alpha \in R$. Since for any $x \in L$ the element $x^{[p]} - x^p$ commutes with L, by Schur's Lemma we have $x^{[p]} - x^p = k(x) \cdot 1_V$ where $k(x) \in \mathbb{F}$. As $\operatorname{rk} 1_V = \dim V > 2^7 d$, $\operatorname{rk} x_\alpha^p \leqslant \operatorname{rk} x_\alpha \leqslant 2^6 d$, and $\operatorname{rk}(h_\alpha - h_\alpha^p) \leqslant \operatorname{rk} h_\alpha \leqslant 2^7 d$, we have $k(x_\alpha) = k(h_\alpha) = 0$ for all $\alpha \in R$, i.e. V is restricted.

(3) It follows from (1) that $r > 2^8$. In particular, L is of type A, B, C, or D. If $p > 3$, then by (2), V is restricted. In particular, V has highest weight and can be constructed from the corresponding highest weight module over $\mathbb{C}$ by reduction modulo p (see [9]). Moreover, V can be lifted to an infinitesimally irreducible representation of the corresponding algebraic group (this obviously holds if $\operatorname{char}\mathbb{F} = 0$). Assume that neither V nor V^* is isomorphic to the standard L-module. Then it follows from [18, Theorem 1.1] that $\dim V \geqslant r^2/2$. Applying (1), we get $r^2/2 \leqslant 2^7 dr$, so $r \leqslant 2^8 d$. Substituting this in (1), we get $\dim V \leqslant 2^{15} d^2$, which is a contradiction. Therefore V is either standard or dual to it, so L can be identified with $\mathfrak{sl}(V)$, $\mathfrak{o}(V, \Phi)$, or $\mathfrak{sp}(V, \Phi)$. □

We remark that after fixing an isomorphism from L onto one of matrix algebras A_n, B_n, C_n, D_n the L-module V becomes the standard module of dimension $n+1$, $2n+1$, $2n$, $2n$, or L is of type A_n and V is the dual of the standard module.

These classification results allow the following important improvement of Proposition 2.7.

Theorem 2.13. *Let $\mathbb{F}$ be an arbitrary field of characteristic $\neq 2, 3$. Suppose G is an irreducible subalgebra of $\mathfrak{gl}(W)$ over $\mathbb{F}$ generated by elements of rank $< d$. Set $\Delta := \mathrm{End}_G W$. Assume that*

$$2^6 57^2 (\dim_{\mathbb{F}} \Delta) d^2 < \dim_{\mathbb{F}} W < \infty.$$

Then G^3 is the unique minimal non-central ideal. Moreover, $G^2 \subset G^3 + C(G)$ is true.

Proof. Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$.

(a) By Wedderburn's theorem, there is a surjective homomorphism

$$U(G) \to \mathrm{End}_\Delta W \cong M(n, \Delta^{\mathrm{opp}}), \quad n := \dim_\Delta W.$$

Thus

$$U(\overline{\mathbb{F}} \otimes_{\mathbb{F}} G) \cong \overline{\mathbb{F}} \otimes_{\mathbb{F}} U(G) \to \overline{\mathbb{F}} \otimes_{\mathbb{F}} M(n, \Delta^{\mathrm{opp}}) \cong M(n, \overline{\mathbb{F}} \otimes_{\mathbb{F}} \Delta^{\mathrm{opp}}).$$

Set $A := \overline{\mathbb{F}} \otimes_{\mathbb{F}} \Delta^{\mathrm{opp}}$ (which is a finite-dimensional $\overline{\mathbb{F}}$–algebra), and $\mathrm{Rad}\, A$ the nilpotent radical. Then $\mathrm{Rad}\, M(n, \overline{\mathbb{F}} \otimes_{\mathbb{F}} \Delta^{\mathrm{opp}}) = M(n, \mathrm{Rad}\, A)$, hence

$$\begin{aligned} M(n, A)/\mathrm{Rad}\, M(n, A) &\cong M(n, A/\mathrm{Rad}\, A) \cong \bigoplus M(n, \bar{A}_i) \\ &\cong \bigoplus_{i=1}^{r} M(n_i, \overline{\mathbb{F}}). \end{aligned}$$

Here we put $A/\mathrm{Rad}\, A = \bigoplus_{i=1}^{r} \bar{A}_i$, where all $\bar{A}_i$ are simple associative $\overline{\mathbb{F}}$-algebras of finite dimension and $n_i = n(\dim_{\overline{\mathbb{F}}} \bar{A}_i)^{1/2}$. Correspondingly,

$$(\overline{\mathbb{F}} \otimes_{\mathbb{F}} W)/(\mathrm{Rad}\, A)(\overline{\mathbb{F}} \otimes_{\mathbb{F}} W) =: \bigoplus_{i=1}^{r} \overline{W}_i$$

decomposes into the direct sum of irreducible $(\overline{\mathbb{F}} \otimes_{\mathbb{F}} G)$-modules. One observes

$$\dim_{\overline{\mathbb{F}}} \overline{W}_i = n_i \geqslant n = \dim_\Delta W = (\dim_{\mathbb{F}} W)/(\dim_{\mathbb{F}} \Delta) > 2^6 57^2 d^2.$$

Let

$$\overline{G} := (\overline{\mathbb{F}} \otimes_{\mathbb{F}} G)/\mathrm{ann}\left(\bigoplus_{k=1}^{r} \overline{W}_k\right)$$

and $\pi : \overline{G} \to \overline{G}/\mathrm{Rad}\, \overline{G}$ the canonical homomorphism. Let $\bigoplus_{i=1}^{s} \widetilde{S}_i$ be the socle of $\pi(\overline{G})$, i.e., the sum of all minimal ideals. Take $S_i' = \pi^{-1}(\widetilde{S}_i) \subset \overline{G}$ the full preimage, set $S_i := S_i'^{(\infty)} \subset \overline{G}$. Then

$$\pi(S_i) = \widetilde{S}_i, \quad S_i^{(1)} = S_i \quad \text{for all } i.$$

Let M be a $\overline{G}$-ideal in S_i. If $\pi(M) = (0)$, then M is solvable. Otherwise $\pi(M) = \widetilde{S}_i$, and hence $S_i = M + (S_i \cap \operatorname{Rad}\overline{G})$. As S_i is perfect, one obtains $M = S_i$. We note as a consequence, that every $\overline{G}$-ideal properly contained in S_i is solvable.

(b) Fix any index i. Suppose k is an index for which $S_i \cdot \overline{W}_k \neq (0)$. Let

$$\varphi_k : \overline{G} \to \mathfrak{gl}(\overline{W}_k)$$

denote the respective representation. As $S_i^{(1)} = S_i$, $\varphi_k(S_i)$ cannot be solvable. We intend to show that

- $\varphi_k(S_i)$ is the unique minimal non-central ideal of $\varphi_k(\overline{G})$,
- $\varphi_k(S_i)$ is simple if $\operatorname{char}\overline{\mathbb{F}} = 0$,
- $\varphi_k(S_i)$ is a Chevalley algebra if $\operatorname{char}\overline{\mathbb{F}} > 3$,
- either $\varphi_k(\overline{G}) = \varphi_k(S_i) \oplus C(\varphi_k(\overline{G}))$ or
  $\varphi_k(\overline{G}) = \varphi_k(S_i) \cong \mathfrak{sl}(m)$ with $\operatorname{char}\overline{\mathbb{F}} \mid m$, or
  $\varphi_k(\overline{G}) \cong \mathfrak{gl}(m)$, $\varphi_k(S_i) \cong \mathfrak{sl}(m)$ with $\operatorname{char}\overline{\mathbb{F}} \mid m$.

Namely, according to Proposition 2.7 $\varphi_k(\overline{G})$ has a unique minimal non-central ideal $\mathcal{J}$, $\overline{W}_k$ is an irreducible $\mathcal{J}$-module and $\mathcal{J}/C(\mathcal{J})$ is simple. Note that the ground field $\overline{\mathbb{F}}$ is algebraically closed. Thus, if $\operatorname{char}\overline{\mathbb{F}} = 0$, then $\mathcal{J}$ is a simple algebra. Moreover, since $\mathcal{J}$ is the unique minimal non-central ideal of $\varphi_k(\overline{G})$, one has $\varphi_k(\overline{G}) = \mathcal{J} \oplus C(\varphi_k(\overline{G}))$. In particular,

$$\varphi_k(S_i) = \varphi_k(S_i)^{(1)} = \mathcal{J}^{(1)} = \mathcal{J}.$$

This is the claim if $\operatorname{char}\overline{\mathbb{F}} = 0$. Suppose $\operatorname{char}\overline{\mathbb{F}} = p > 3$. Recall that $\overline{W}_k$ is an irreducible $\mathcal{J}$-module of dimension $> 2^6 57^2 d^2 > 2^{11} d^2$. Corollary 2.11 now shows that $\mathcal{J}/C(\mathcal{J})$ cannot be of Cartan or Melikian type. The classification Theorem 2.8 then yields that $\mathcal{J}/C(\mathcal{J})$ is classical simple. By [7], either the central extension splits or $\mathcal{J} \cong \mathfrak{sl}(m) \oplus C$ with $C \subset C(\mathcal{J})$ and $p \mid m$.

The minimality of $\mathcal{J}$ implies $\mathcal{J}^{(1)} = \mathcal{J}$, hence either $C(\mathcal{J}) = (0)$ or $\mathcal{J} \cong \mathfrak{sl}(m)$, $p \mid m$.

Finally suppose $\mathcal{J} \cong \mathfrak{psl}(m)$, $p \mid m$. Put in Theorem 2.12 $L = \mathfrak{sl}(m)$, and $V = \overline{W}_k$ with representation given by that of $\mathcal{J}$. Theorem 2.12(3) shows that the action of L is faithful, which means that this case is impossible. As a result, $\mathcal{J}$ is a Chevalley algebra in all cases.

Let $K := \{x \in \varphi_k(\overline{G}) \mid [x, \mathcal{J}] = (0)\}$. Since $\mathcal{J}$ is the unique minimal non-central ideal and K is an ideal, one has $K = C(\varphi_k(\overline{G}))$. Suppose $\mathcal{J} = \mathfrak{sl}(m)$ with $p \mid m$. Then

$$(0) \neq C(\mathcal{J}) \subset C\big(\varphi_k(\overline{G})\big).$$

Since $\overline{W}_k$ is $\varphi_k(\overline{G})$-irreducible, one has $\dim C(\varphi_k(G)) \leqslant 1$. Hence $C(\varphi_k(\overline{G})) \subset \mathcal{J}$ in this case. Therefore either

$$\varphi_k(\overline{G}) \cong \mathfrak{gl}(m) \quad \text{or} \quad \varphi_k(\overline{G}) \cong \mathfrak{sl}(m).$$

In all other cases $\operatorname{Der}\mathcal{J} \cong \operatorname{ad}\mathcal{J}$ holds, whence

$$\varphi_k(\overline{G}) = \mathcal{J} \oplus C\big(\varphi_k(\overline{G})\big).$$

Consequently, $\varphi_k(S_i) = \varphi_k(S_i)^{(1)} = \mathcal{J}$ in all cases. This finishes the proofs of all claims.

(c) Recall that by definition $\widetilde{S}_i \cong S_i/S_i \cap (\operatorname{rad}\overline{G})$, and $\widetilde{S}_i$ is $\overline{G}$-simple. Let k be as in (b), so that $\varphi_k(S_i) \neq (0)$. Then it is only possible that $S_i \cap (\ker\varphi_k) \subset \operatorname{rad}\overline{G}$ (see (a)). Therefore $\widetilde{S}_i$ is a non-solvable homomorphic image of $\varphi_k(S_i)$. Since it is $\overline{G}$-simple, the previous results on $\varphi_k(S_i)$ show that $\widetilde{S}_i$ is classical simple.

Consequently, the socle $\bigoplus_{i=1}^{s}\widetilde{S}_i$ of $\pi(\overline{G})$ is the direct sum of classical simple algebras. Since $\pi(\overline{G})$ maps injectively via the ad-representation into the derivation algebra of its socle, we obtain that

$$\pi(\overline{G})^{(1)} \subset \bigoplus_{i=1}^{s}\widetilde{S}_i.$$

This in turn means

$$\overline{G}^{(1)} \subset \sum_{i=1}^{s} S_i + \operatorname{rad}\overline{G}.$$

(d) Due to Proposition 2.7 G has a unique minimal non-central ideal D. Let $\overline{D}$ denote the image of $\overline{\mathbb{F}} \otimes_{\mathbb{F}} D$ in $\overline{G}$. Then $\overline{D}$ is a perfect algebra, as D is so. Hence

$$\overline{D} = \overline{D}^{(\infty)} \subset \left(\sum_{i=1}^{s} S_i + \operatorname{rad}\overline{G}\right)^{(\infty)} = \sum_{i=1}^{s} S_i.$$

Suppose there is $i \leqslant s$ such that $S_i \not\subset \overline{D}$. Since $\overline{G}$ acts faithfully on $\bigoplus \overline{W}_k$, there is k, so that $\varphi_k(S_i) \neq (0)$. Due to (b), $\varphi_k(S_i)$ is the unique minimal non-solvable ideal of $\varphi_k(\overline{G})$. Thus either $\varphi_k(\overline{D}) \subset C(\varphi_k(\overline{G}))$ or $\varphi_k(S_i) \subset \varphi_k(\overline{D})$. In the first case the perfectness of $\overline{D}$ implies $\varphi_k(\overline{D}) = (0)$. In the second case one has $[\varphi_k(S_i), \varphi_k(\overline{D})] \supset \varphi_k(S_i)^{(1)} = \varphi_k(S_i)$. But $[S_i, \overline{D}]$ is a proper $\overline{G}$-ideal in S_i, hence is solvable (see (a)). Consequently, the latter case does not occurs. Then

$$\overline{D}\Big(\bigoplus \overline{W}_j\Big) \subset \bigoplus_{j\neq k} \overline{W}_j.$$

This in turn means that

$$(\overline{\mathbb{F}} \otimes_{\mathbb{F}} D)(\overline{\mathbb{F}} \otimes_{\mathbb{F}} W) \neq \overline{\mathbb{F}} \otimes_{\mathbb{F}} W.$$

However, W is D-irreducible. This contradiction proves that $S_i \subset \overline{D}$ for all i. Consequently

$$\overline{D} = \sum_{i=1}^{s} S_i.$$

(e) Combining the results of (c) and (d) one obtains

$$\overline{G}^{(1)} \subset \overline{D} + \operatorname{rad} \overline{G}.$$

Note that $\operatorname{ann}_{\overline{\mathbb{F}} \otimes_{\mathbb{F}} G}(\bigoplus \overline{W}_j)$ is mapped under the homomorphism $U(\overline{\mathbb{F}} \otimes_{\mathbb{F}} G) \to M(n, \overline{\mathbb{F}} \otimes_{\mathbb{F}} \Delta^{\mathrm{opp}})$ into $M(n, \operatorname{Rad} A)$. Therefore it acts nilpotently on $\overline{\mathbb{F}} \otimes_{\mathbb{F}} W$. We now get

$$\overline{\mathbb{F}} \otimes_{\mathbb{F}} G^{(1)} = (\overline{\mathbb{F}} \otimes_{\mathbb{F}} G)^{(1)} \subset (\overline{\mathbb{F}} \otimes_{\mathbb{F}} D) + \operatorname{rad}(\overline{\mathbb{F}} \otimes_{\mathbb{F}} G).$$

Since $(\operatorname{rad}(\overline{\mathbb{F}} \otimes_{\mathbb{F}} G)) \cap (1 \otimes G) \subset 1 \otimes (\operatorname{rad} G)$, this gives $G^{(1)} \subset D + \operatorname{rad} G$. Proposition 2.7 shows that $\operatorname{rad} G = C(G)$. Consequently, $G^3 = [D, G] = D$. This proves the theorem. □

3. Local systems

The method of “small rank generators” developed in Section 2 will now be combined with the beautiful ideas of [14–17]. This procedure will simplify the proofs of [14–17], as well as will provide similarly complete results in positive characteristic. Let in this section L denote an *irreducible* subalgebra of $\mathfrak{fgl}(V)$ where V is an *infinite*-dimensional vector space over an *arbitrary* field $\mathbb{F}$.

Definition. A family Π of subalgebras of L is called a *local system* if

(a) for $H_1, H_2 \in \Pi$ there exists $H_3 \in \Pi$ with $H_1 + H_2 \subset H_3$;
(b) $\bigcup\{H \in \Pi\} = L$.

Since L is locally finite, the family of all finite-dimensional subalgebras is a local system. In this section we are going to construct a nicely behaving local system for L by refining this system.

The following statement is well known.

Lemma 3.1. *The centralizer* $\Delta := \operatorname{End}_L V$ *of* L *in* $\operatorname{End}_{\mathbb{F}} V$ *is a finite-dimensional division algebra over* $\mathbb{F}$. *Moreover,* $\dim_{\mathbb{F}} \Delta \leqslant \min\{\operatorname{rk} x \mid x \in L \setminus (0)\}$.

Proof. By Schur’s lemma, Δ is a division algebra. Observe that $\delta x V = x V$ for all $\delta \in \Delta$ and $x \in L$. Therefore, xV is Δ-invariant, i.e. a finite-dimensional Δ-space, so $\dim_{\mathbb{F}} \Delta \mid \operatorname{rk} x$.

Lemma 3.2. *Let* K *be a finite-dimensional subalgebra of* L *and let* V_0 *be a finite-dimensional subspace of* V. *Then there exists a finite-dimensional subalgebra* $\mathcal{L}(K, V_0)$ *containing* K *such that for every* $\mathcal{L}(K, V_0)$*-submodule* M *of* V *one has either* $(KV + V_0) \cap M = (0)$ *or* $KV + V_0 \subset M$. *In particular,* $KV + V_0 \subset \mathcal{L}(K, V_0)V$.

Proof. Let $\Delta = \mathrm{End}_L V$. Then by Lemma 3.1, $\dim_{\mathbb{F}} \Delta < \infty$. Recall that KV is finite-dimensional. Therefore $W = \Delta(KV + V_0)$ is a finite-dimensional Δ-space. Since $U(L)$ acts irreducibly on V, by Jacobson's Density Theorem there is a subalgebra A of $U(L)$ such that W is an irreducible A-module (with the same centralizer Δ). As $\dim_{\mathbb{F}} W \leqslant \dim_{\mathbb{F}} \Delta \cdot \dim_{\mathbb{F}}(KV + V_0) < \infty$, we can assume that A is finitely generated. Therefore there exists a finitely generated subalgebra $\mathcal{L}(K, V_0)$ of L containing K such that $A \subset U(\mathcal{L}(K, V_0))$. It remains to observe that $\mathcal{L}(K, V_0)$ is finite-dimensional (since L is locally finite) and for every $\mathcal{L}(K, V_0)$-submodule M of V the space $M \cap W$ is an A-submodule of W. $\square$

Lemma 3.3. (1) *If* $\dim V_0 \geqslant 2$ *then* $\mathcal{L}(K, V_0)$ *is not nil.*

(2) *Let K be not nil and $h \in K$ a non-nilpotent element. If* $\dim V_0 > \mathrm{rk}\, h$*, then* $\mathcal{L}(K, V_0)/\operatorname{nil} \mathcal{L}(K, V_0)$ *is not nilpotent.*

Proof. (1) Suppose $\mathcal{L}(K, V_0)$ is nil. By Engel's theorem every composition series of $\mathcal{L}(K, V_0)V$ is a flag. So all factors are 1-dimensional. This contradicts Lemma 3.2.

(2) Set $\mathcal{L} = \mathcal{L}(K, V_0)$. Suppose $\mathcal{L}/\operatorname{nil} \mathcal{L}$ is nilpotent. By Lemma 3.2, there are $\mathcal{L}$-submodules $M \supset N$ of $\mathcal{L}V$ such that $M \supset KV + V_0$, $N \cap (KV + V_0) = (0)$ and $W = M/N$ is irreducible. Observe that $\dim W \geqslant \dim V_0$ and $(\operatorname{nil} \mathcal{L})W = (0)$.

If $\mathrm{char}\, \mathbb{F} = 0$, then Lie's theorem shows that $\mathcal{L}^{(1)} \subset \operatorname{nil} \mathcal{L}$. As W is $\mathcal{L}$-irreducible, this implies that W is a $\mathcal{L}/\mathcal{L}^{(1)}$-module. Therefore hW is $\mathcal{L}$-invariant. Since $\mathrm{rk}\, h < \dim W$, we have $hW \neq W$. Then $hW = 0$ and $h^3 V \subset h^2 M \subset hN = (0)$. This implies that h is nil, a contradiction.

If $\mathrm{char}\, \mathbb{F} = p > 0$, then there is n such that $[h^{p^n}, \mathcal{L}] \subset \operatorname{nil} \mathcal{L}$ (as $\mathcal{L}/\operatorname{nil} \mathcal{L}$ is nilpotent). Since $\mathrm{rk}\, h^{p^n} \leqslant \mathrm{rk}\, h < \dim W$, we derive a contradiction as before. $\square$

For any finite-dimensional subalgebra K of L set $V_K := KV/\operatorname{ann}_{KV} K$, which is a finite-dimensional K-module. Observe that, if $x \in K$ annihilates V_K, then $x^3 V = 0$. Assume that V_K is irreducible. Then combining this with the Remark of Section 2 shows that

$$\operatorname{ann}_K V_K = \operatorname{nil} K.$$

According to Lemma 3.3 there exists a finite-dimensional subalgebra Q of L containing an element h_0 such that $\mathrm{ad}_{Q/\operatorname{nil} Q}\, h_0$ is not nilpotent. Then

$$Q \cap L^1(\mathrm{ad}\, h_0) \not\subset \operatorname{nil} Q.$$

Therefore the ideal of Q generated by $Q \cap L^1(\mathrm{ad}\, h_0)$ is not a nil ideal. Set $d_0 := \mathrm{rk}\, h_0$.

Lemma 3.4. *Let K be a finite-dimensional subalgebra of L containing Q, and $m \in \mathbb{N}$. Then there exists a finite-dimensional subalgebra K_1 such that the following is true*:

(1) $K \subset K_1$;
(2) V_{K_1} *is an irreducible* K_1*-module*;
(3) $\dim V_{K_1} > m$.

Proof. (a) Let U denote any subspace of V of dimension $m+1$, set according to Lemma 3.2

$$H := \mathcal{L}(K, U) \quad \text{and} \quad K_1 \text{ the ideal of } \mathcal{L}(H, U) \text{ generated by } H.$$

Note that $K \subset H \subset K_1$. Let

$$\mathcal{L}(H, U)V =: V_0 \supset V_1 \supset \cdots \supset V_r \supset (0)$$

be a $\mathcal{L}(H, U)$-composition series of V_0 and $k \geqslant 0$ such that $HV \subset V_k$, $HV \not\subset V_{k+1}$. By definition we obtain $HV \cap V_{k+1} = (0)$. The space $\{h \in K_1 \mid hV \subset V_k\}$ contains H and it is an ideal of $\mathcal{L}(H, U)$. Hence it coincides with K_1. Therefore $K_1 V \subset V_k$ holds. By the same argument we obtain $K_1 V_{k+1} = (0)$.

We next consider a K_1-composition series

$$V_k = V_{k,0} \supset V_{k,1} \supset \cdots \supset V_{k,s} = V_{k+1}.$$

Note that, since V_k/V_{k+1} is $\mathcal{L}(H, U)$-irreducible and K_1 is an ideal in $\mathcal{L}(H, U)$, all composition factors $V_{k,i}/V_{k,i+1}$ are isomorphic (Lemma 2.5). Observe that these factors are isomorphic as K-modules, as $K \subset K_1$. If for some i, $KV \cap V_{k,i} \neq (0)$, then the definition of H implies that $KV \subset V_{k,i}$. Suppose V_k/V_{k+1} is not K_1-irreducible, i.e. $s \geqslant 2$. If $KV \cap V_{k,s-1} \neq (0)$, then $KV \subset V_{k,s-1}$ and hence $K(V_{k,0}/V_{k,1}) = (0)$. If $KV \cap V_{k,s-1} = (0)$, then $K(V_{k,s-1}/V_{k,s}) = (0)$. Thus in either case K annihilates one of the factors, hence all. Therefore K acts nilpotently on V_k/V_{k+1}. As $KV \subset V_k$ and $KV_{k+1} = (0)$, K acts nilpotently on V as well. But then $Q \subset K$ is nil, contrary to our assumption. Thus V_k/V_{k+1} is K_1-irreducible.

Let $\pi : V_k \to V_k/V_{k+1}$ denote the canonical homomorphism. Since V_k/V_{k+1} is K_1-irreducible and $\pi(K_1 V) \neq (0)$ (by the choice of k) we have that $\pi(K_1 V)$ is K_1-irreducible. Since $\ker \pi = V_{k+1} \subset \operatorname{ann}_V K_1$, this yields that $V_{K_1} = K_1 V/\operatorname{ann}_{K_1 V} K_1 \cong V_k/V_{k+1}$ is K_1-irreducible.

Recall that $HV \subset V_k$, $HV \cap V_{k+1} = (0)$. As $U \subset HV$, there is an embedding $U \to V_k/V_{k+1}$. Then $\dim V_{K_1} > m$. $\square$

Applying this lemma twice one obtains the following lemma.

Lemma 3.5. *Let K be a finite-dimensional subalgebra of L containing Q and $m \in \mathbb{N}$. Then there exists a finite-dimensional subalgebra K_2 such that the following is true.*

(1) $K \subset K_2$.
(2) $KV \cap \operatorname{ann}_V K_2 = 0$.

(3) *K_2 is generated by elements of rank* $\leqslant 2\max\{\operatorname{rk} x \mid x \in K\}$.
(4) *V_{K_2} is an irreducible K_2-module.*
(5) $\dim V_{K_2} > m$.

Proof. (a) Let K_1 and K_2' be the algebras constructed according to Lemma 3.4 with respect to K and m, respectively K_1 and $m_1 := 2dd_1 + 1$, where $d := \max\{\operatorname{rk} x \mid x \in K\}$ and $d_1 := \max\{\operatorname{rk} x \mid x \in K_1\}$. This implies that $K \subset K_1 \subset K_2'$, V_{K_1} and $V_{K_2'}$ are K_1- and K_2'-irreducible, $\dim V_{K_1} > m$, $\dim V_{K_2'} > \max\{m, 2dd_1\}$.

(b) Let I denote the subalgebra of K_2' generated by $K_2' \cap L^1(\operatorname{ad} h_0)$. This is an ideal of K_2' (Lemma 2.1(3)) and it is not nil (by the choice of Q and h_0). Set in Lemma 2.6 $G = K_2'/\operatorname{nil} K_2'$, $S = I + \operatorname{nil} K_2'/\operatorname{nil} K_2'$, $H = K_1 + \operatorname{nil} K_2'/\operatorname{nil} K_2'$. Then $d^H \leqslant d_1$ and

$$d_S \leqslant \max\bigl\{\operatorname{rk} u \bigm| u \in (K_2') \cap L^1(\operatorname{ad} h_0)\bigr\} \leqslant 2\operatorname{rk} h_0 \leqslant 2d.$$

That lemma proves that every I-submodule of $V_{K_2'}$ is K_1-invariant. Since V_{K_1} is K_1-irreducible, there is only one non-trivial K_1-composition factor of $V_{K_2'}$. Suppose $V_{K_2'}$ is not I-irreducible. Then K_1 acts trivially on one of the I-factors. Thus $I \cap K_1$ acts trivially on such a factor. By Lemma 2.5 all I-factors of $V_{K_2'}$ are isomorphic. But then $I \cap K_1$ acts nilpotently on V. We now observe that $Q^1(\operatorname{ad} h_0) \subset I \cap K_1$ is not nil by the choice of h_0. This contradiction shows that $V_{K_2'}$ is I-irreducible.

(c) Set $K_2 := K + I \subset K_2'$. Then (1) and (3) are true by construction. Let $K_2V \to V_{K_2'}$ denote the canonical homomorphism. Since $V_{K_2'}$ is I-irreducible, we obtain an isomorphism of K_2-modules

$$V_{K_2'} \cong K_2V/\operatorname{ann}_{K_2V} K_2' \cong K_2V/\operatorname{ann}_{K_2V} K_2 = V_{K_2}.$$

This proves (4), (5). It also shows that $\operatorname{ann}_{K_2V} K_2 \subset \operatorname{ann}_{K_2'V} K_2'$. Thus

$$KV \cap \operatorname{ann}_V K_2 \subset (K_1V) \cap \operatorname{ann}_{K_2'V} K_2' = (0).$$

This proves (2). □

Lemma 3.6. *Let G be a finite-dimensional Lie algebra, W a finite-dimensional G-module such that GW is G irreducible. Let $K \subset G$ be a subalgebra such that*

$$W = \operatorname{ann}_W K + GW.$$

Then there is a subalgebra G_0 of G satisfying

$$K \subset G_0, \qquad G_0 + \operatorname{nil}_W G = G, \qquad \operatorname{nil}_W G_0 = \operatorname{ann}_{G_0} W.$$

Proof. (a) Let G_0 be a subalgebra of G such that $K \subset G_0$, $G_0 + \operatorname{nil}_W G = G$, and $\dim G_0$ is minimal subject to these conditions. Set

$$R := \operatorname{nil}_W G_0$$

and assume that $R \neq \mathrm{ann}_{G_0} W$. Note that $R + \mathrm{nil}_W G$ is an ideal of G which acts nilpotently on W. Hence $R + \mathrm{nil}_W G = \mathrm{nil}_W G$, and

$$R \subset \mathrm{nil}_W G. \tag{2}$$

In particular, $R(GW) = (0)$.

(b) By the present assumption, $(0) \neq RW \subset GW$. Observe that $RW = R(\mathrm{ann}_W K)$, as $R(GW) = (0)$. Fix $w \in \mathrm{ann}_W K$ with $Rw \neq (0)$. Note that GW is in fact $(G/\mathrm{nil}_W G)$-irreducible, hence it is G_0-irreducible. Now

$$G_0(Rw) \subset [G_0, R]w + R(G_0 w) \subset Rw + (0).$$

Hence Rw is G_0-invariant. As GW is G_0-irreducible, we obtain $Rw = GW$. Thus for every $g \in G_0$ there is $r \in R$ such that $gw = rw$. Set

$$G' := \{g \in G_0 \mid gw = 0\}.$$

It is shown that $G' + R = G_0$. By (2), this implies $G' + \mathrm{nil}_W G = G_0 + \mathrm{nil}_W G = G$. Clearly, $K \subset G'$. The minimality of G_0 implies $G_0 = G'$, a contradiction. □

Lemma 3.7. *Let K be a finite-dimensional subalgebra of L containing Q, and $m \in \mathbb{N}$. Then there exist $d = d(K) \in \mathbb{N}$ and a finite-dimensional subalgebra H, which satisfy the following.*

(1) $K \subset H$.
(2) $\dim V_H > m$.
(3) $(\mathrm{nil}\, H)HV = H(\mathrm{nil}\, H)V = (0)$.
(4) *$H/\mathrm{nil}\, H$ has a unique minimal non-central ideal S. The algebra $S/C(S)$ is simple, V_H is S-irreducible.*
(5) *The image of H in $\mathfrak{gl}(V_H)$ is generated by elements of rank $< d(K)$.*

Proof. Note that $K/\mathrm{nil}\, K$ is not nilpotent as $Q \subset K$ has this property.

(a) Fix a finite-dimensional subspace $V' \subset V$ such that

$$\mathrm{ann}_V K \oplus V' = V.$$

Set

$$d := d(K) := 2 \max\bigl\{\mathrm{rk}\, u \bigm| u \in \mathcal{L}(K, V')\bigr\}.$$

Observe that d does depend on K (and any choice of V') only. Let K_2 be an algebra associated to $\mathcal{L}(K, V')$ and $\max\{m, 8d^2\}$ according to Lemma 3.5. Then K_2 is generated by elements of rank $\leqslant d$ and $\dim V_{K_2} > m$. Since $V' \subset \mathcal{L}(K, V')V \subset K_2 V$, one has

$$\mathrm{ann}_V K + K_2 V = V.$$

(b) Let H be a subalgebra minimal subject to the conditions

$$K \subset H \subset K_2, \qquad H + \mathrm{nil}\, K_2 = K_2.$$

As V_{K_2} is K_2-irreducible, $\operatorname{nil} K_2$ annihilates V_{K_2}. Therefore V_{K_2} is H-irreducible. The K_2-homomorphism $\lambda : K_2V \to V_{K_2}$ maps HV onto a non-zero H-submodule of V_{K_2}. Thus $\lambda(HV) = V_{K_2}$. Hence $HV + \operatorname{ann}_{K_2V} K_2 = K_2V$.

As a first consequence we observe that the H-submodule $\lambda(\operatorname{ann}_{HV} H)$ of V_{K_2} has to be zero. Therefore $\operatorname{ann}_{HV} H = \ker(\lambda) \cap HV$ holds, and λ gives rise to an isomorphism $V_H \cong V_{K_2}$. Thus $\dim V_H > m$. In particular, (1), (2) hold.

As a second consequence, $HV + \operatorname{ann}_{K_2V} K_2 = K_2V$. Since $K \subset K_2$ this gives

$$V = \operatorname{ann}_V K + K_2V = \operatorname{ann}_V K + HV. \tag{3}$$

(c) Apply Lemma 3.6 with

$$G = H, \qquad W = V/\operatorname{ann}_V H.$$

Then there is a subalgebra $H' \subset H$ with

$$K \subset H', \qquad H' + \operatorname{nil}_W H = H, \qquad \operatorname{nil}_W H' = \operatorname{ann}_{H'} W.$$

Clearly, $\operatorname{nil}_W H = \operatorname{nil} H$. As $K_2 = H + \operatorname{nil} K_2$, $\operatorname{nil} H + \operatorname{nil} K_2$ is an ideal of K_2, whence $\operatorname{nil} H + \operatorname{nil} K_2 = \operatorname{nil} K_2$ and $\operatorname{nil} H \subset \operatorname{nil} K_2$. Thus $H' + \operatorname{nil} K_2 = K_2$. The minimality of H proves $H' = H$. Then

$$(\operatorname{nil} H)V \subset \operatorname{ann}_V H.$$

Next apply Lemma 3.6 with

$$G = H, \qquad W := (HV)^*.$$

The series of H-modules

$$HV \supset \operatorname{ann}_{HV} H \supset (0)$$

gives rise to the dual sequence

$$(0) \subset HW \subset W$$

where $HW \cong (V_H)^*$ is H-irreducible. We mentioned above that $\operatorname{ann}_{HV} K_2 = \operatorname{ann}_{HV} H$. Thus the definition of K_2 yields $KV \cap \operatorname{ann}_V H \subset KV \cap \operatorname{ann}_{K_2V} K_2 = (0)$. We now have the dual relation

$$\operatorname{ann}_W K + HW = W.$$

As above Lemma 3.6 yields $(\operatorname{nil} H)W = (0)$, whence

$$(\operatorname{nil} H)HV = (0).$$

This proves (3).

(d) Set in Proposition 2.7

$$G = K_2/\operatorname{nil} K_2 \cong H/\operatorname{nil} H, \qquad W = V_H \cong V_{K_2}.$$

By Lemma 3.4(3) K_2 is generated by elements of rank $\leqslant d$. Since $\dim W > 8d^2$, there is a unique minimal non-central ideal S of G and $S^{(1)} = S$, $S/C(S)$ is simple, W is S-irreducible. This proves (4).

(e) Since $H + \operatorname{nil} K_2 = K_2$ and K_2 is generated by elements of rank $< d$, (5) follows. □

Lemma 3.8. *Let $G \subset \mathfrak{fgl}(V)$ be a finite-dimensional subalgebra and $A \subset \operatorname{End} V$ the associative algebra generated by G. Suppose V_G is G-irreducible. If there is $z \in A$ such that*

(a) $\operatorname{ad}_A z$ *is locally nilpotent,*
(b) z *is not nilpotent,*

then V splits as a G-module into the direct sum of its Fitting components with respect to z,

$$V = V^0(z) \oplus V^1(z),$$

and $G(GV^0(z)) = (0)$, $V^1(z) \cong V_G$ holds.

Proof. Since A consists of elements of finite rank, V allows a Fitting decomposition with respect to z:

$$V = V^0(z) \oplus V^1(z).$$

By assumption (a), for any $a \in A$, there is $k > 0$ such that $(\operatorname{ad} z)^k(a) = 0$. Then $V^0(z)$ and $V^1(z)$ are invariant under all $a \in A$, i.e., these are G-submodules. Since z acts invertibly on $V^1(z)$, we have

$$V^1(z) = zV^1(z) \subset GV, \qquad V^1(z) \cap \operatorname{ann}_V G = (0).$$

Thus there is an injective G-module homomorphism $V^1(z) \to V_G$. Since z is not locally nilpotent, $V^1(z) \neq (0)$. As V_G is G-irreducible, $V^1(z) \cong V_G$. Note that $GV \supset V^0(z) \cap GV \supset \cdots$ can be extended to a composition series. By the Jordan–Hölder Theorem, all composition series of GV have only one non-trivial composition factor. As $V^1(z)$ injects into $GV/V^0(z) \cap GV$, this is only possible if G acts nilpotently on $V^0(z) \cap GV$. Then $V^0(z) \cap GV$ maps trivially into the irreducible G-module $GV/\operatorname{ann}_{GV} G = V_G$. Hence $V^0(z) \cap GV \subset \operatorname{ann}_{GV} G$, and therefore $G(GV^0(z)) = (0)$. □

Lemma 3.9. *Let $H \subset \mathfrak{fgl}(V)$ be a finite-dimensional subalgebra satisfying*

(a) V_H *is H-irreducible,*
(b) $(\operatorname{nil} H)HV = H(\operatorname{nil} H)V = (0)$,
(c) $H/\operatorname{nil} H$ *has a unique minimal non-central ideal S, and $S/C(S)$ is simple.*

Then

(1) $\operatorname{rad} H = C(H)$,

(2) *H has a unique minimal non-central ideal S_H, $S_H/(C(H)\cap S_H)$ is simple and $S_H/(\operatorname{nil} H)\cap S_H \cong S$.*

Proof. (1) One has that $[H, \operatorname{nil} H]$ annihilates V. Hence $\operatorname{nil} H \subset C(H)$. If $\operatorname{rad} H = \operatorname{nil} H$, then $\operatorname{rad} H = C(H)$. Suppose $\operatorname{rad} H \neq \operatorname{nil} H$. Then there is an ideal J of H such that $J^{(1)} \subset \operatorname{nil} H$ but $J \not\subset \operatorname{nil} H$. Choose $z \in J \setminus \operatorname{nil} H$. Then $(\operatorname{ad} z)^3(H) \subset [H, \operatorname{nil} H] = (0)$. Thus Lemma 3.8 applies with $G = H$. According to this lemma there is a decomposition $V = V^0(z) \oplus V^1(z)$ and

$$\big([H, \operatorname{rad} H]\big)\big(V^0(z)\big) = (0).$$

Set $\bar{\ }: H \to H/\operatorname{nil} H$ the canonical homomorphism. If $\operatorname{rad} \overline{H}$ is not central then it contains the unique minimal non-central ideal S. As $S^{(1)} = S$, this is impossible. Hence $[\overline{H}, \operatorname{rad} \overline{H}] = (0)$, which means that $[H, \operatorname{rad} H] \subset \operatorname{nil} H$. But then $[H, \operatorname{rad} H]$ annihilates V_H, and by Lemma 3.7 it annihilates $V^1(z)$. Then it annihilates all of V, whence $[H, \operatorname{rad} H] = (0)$.

(2) Let S' be the full preimage of S in H and $S_H := S'^{(\infty)}$. Note that $\overline{S_H} = \overline{S'}^{(\infty)} = S^{(\infty)} = S$. Let J be any non-central ideal of H. If $\overline{J}$ is central in $\overline{H}$, then $J \subset \operatorname{rad} H = C(H)$, a contradiction. Hence $\overline{J}$ contains S. But then $S' \subset J + \operatorname{rad} H$, whence $S_H = S'^{(\infty)} \subset J$. Therefore S_H is the unique minimal non-central ideal of H. By construction one has $S = \overline{S_H} \cong S_H/(\operatorname{nil} H) \cap S_H$. Let R denote the full preimage of $C(S)$ in H. Since $C(S)$ is an $\overline{H}$-ideal, R is an ideal of H. Moreover, as $\overline{\operatorname{rad} S_H} \subset \operatorname{rad} \overline{S_H} \subset C(S)$ one has $\operatorname{rad} S_H \subset R$. Note that R is solvable, hence $R \subset C(H)$. Thus

$$S_H/C(H) \cap S_H \cong S_H/R \cap S_H \cong S/C(S)$$

is simple. $\square$

The first main result of this section describes a suitable local system over arbitrary fields.

Theorem 3.10. *Let $\mathbb{F}$ be any field, V an infinite-dimensional $\mathbb{F}$-vector space, and $L \subset \mathfrak{fgl}(V)$ an irreducible subalgebra. Fix a finite-dimensional subalgebra Q of L such that $Q/\operatorname{nil} Q$ is not nilpotent, and $h_0 \in Q$ for which $\operatorname{ad}_{Q/\operatorname{nil} Q} h_0$ is not nilpotent. Set $d_0 := \operatorname{rk} h_0$. For every $n \in \mathbb{N}$ the family of all finite-dimensional subalgebras H of L fulfilling the following conditions is a local system $\mathcal{S}(Q, n)$ of L.*

(a) $\operatorname{rad} H = C(H)$.
(b) $(\operatorname{nil} H)HV = H(\operatorname{nil} H)V = (0)$.
(c) $\dim V_H > n$.
(d) *H has a unique minimal non-central ideal S_H, and S_H satisfies*
  (i) $S_H^{(1)} = S_H$,
  (ii) *$S_H/C(H) \cap S_H$ is simple,*

(iii) S_H *is spanned by elements of rank* $\leqslant 4d_0$,
(iv) V_H *is* S_H*-irreducible.*
(e) *If* $\operatorname{char}\mathbb{F} \neq 2, 3$, *then* $S_H = H^3$.

Proof. According to Lemma 3.3 there is a subalgebra $Q \subset L$ having the mentioned properties. The family of all finite-dimensional subalgebras of L containing Q is a local system of L. Let K be an algebra of this local system, $d(K)$ a natural number mentioned in Lemma 3.7, and $m \geqslant \max\{n, 2^6 57^2 d_0 d(K)^2\}$. Choose a subalgebra H associated to K and m according to Lemma 3.7. Note that $K \subset H$.

(a) Lemma 3.9 applies and yields $\operatorname{rad} H = C(H)$.

(b), (c) hold according to the assumption on H.

(d) The existence of S_H and (i), (ii) follow immediately from Lemma 3.9(2).

(iii) Set

$$I := H^1(\operatorname{ad} h_0) + \big[H^1(\operatorname{ad} h_0), H^1(\operatorname{ad} h_0)\big].$$

This ideal of H contains $Q^1(\operatorname{ad} h_0) \neq (0)$, and it is generated by elements of rank $\leqslant 2d_0$ (Lemma 2.2). It is not contained in $C(H)$ as $[h_0, I] \neq (0)$. Thus it contains S_H. Then $S_H = S_H^{(1)} \subset [I, S_H] \subset S_H$. Since $[I, S_H]$ is spanned by elements of rank $\leqslant 4d_0$, the assertion follows.

(iv) follows from the fact, that V_H is S-irreducible (Lemma 3.7(4)), $(\operatorname{nil} H)V_H = (0)$, and $S \cong S_H/(\operatorname{nil} H) \cap S_H$ (Lemma 3.9(2)).

(e) Suppose $\operatorname{char}\mathbb{F} \neq 2, 3$. Set in Theorem 2.13 $W := V_H$, $G \cong \overline{H}$ the image of H in $\mathfrak{gl}(W)$. By Lemma 3.7, G is spanned by elements of rank $< d(K)$.

Set $\Delta := \operatorname{End}_G W$. Then Δ is a division algebra. Since $h_0 \notin \operatorname{nil} H$, one has $h_0 W \neq (0)$. Since $h_0 W$ is a non-zero Δ-vector space, it follow that

$$\dim_{\mathbb{F}} \Delta \leqslant \dim_{\mathbb{F}} h_0 \cdot W \leqslant \operatorname{rk} h_0 = d_0.$$

Therefore the assumptions of Theorem 2.13 are satisfied. As a consequence,

$$S = \overline{H}^3, \qquad \overline{H}^2 \subset \overline{H}^3 + C(\overline{H}).$$

By construction of S_H and (a) this implies

$$S_H + C(H) = H^3 + C(H), \qquad H^2 + C(H) = H^3 + C(H).$$

Then

$$S_H = [S_H, H] = \big[H^3 + C(H), H\big] = \big[H^2 + C(H), H\big] = H^3.$$

We have now proved that every algebra K of the initial local system is contained in an algebra H satisfying (a)–(e). Then the family of these latter algebras form a local system. □

For algebraically closed fields Theorem 3.10 can be improved. The best we can prove in characteristic 0 is the following theorem.

Theorem 3.11. *Let $\mathbb{F}$ be an algebraically closed field of characteristic* 0*, V an infinite-dimensional $\mathbb{F}$-vector space, and $L \subset \mathfrak{fgl}(V)$ an irreducible subalgebra. There is $X \in \{A, B, C, D\}$ such that the following is true. The family of all finite-dimensional subalgebras H of L fulfilling the following conditions is a local system $\mathcal{X}$ of L.*

(a) $H = H^{(1)} \oplus C(H)$.
(b) $H^{(1)} \cong X_{k(H)}$ *($k(H) \geqslant 4$) is a simple algebra of classical type X.*
(c) $\operatorname{nil} H = (0)$.
(d) $H^{(1)}$ *is spanned by elements of rank* 2.
(e) $V = (\operatorname{ann} H) \oplus (HV)$.
(f) *HV is $H^{(1)}$-irreducible, and is the natural module of respective dimension $k(H)+1$, $2k(H)+1$, $2k(H)$, $2k(H)$ for $X = A, B, C, D$.*

Proof. (i) Let Q, h_0, d_0 be as before, and

$$m > 2^{19} d_0^2.$$

Let $H \in \mathcal{S}(Q, m)$. By (a) of Theorem 3.10, $H = H^{(1)} \oplus C(H)$ and $H^{(1)}$ is semi-simple. Due to (d)(i), $S_H = H^{(1)}$ is simple. Since V_H is an irreducible S_H-module of dimension $> 2^{19} d_0^2$ and S_H is spanned by elements of rank $\leqslant 4d_0$, Theorem 2.12 shows that the rank of S_H exceeds $2^{10} d_0$. Therefore S_H occurs in one of the classical series $A - D$.

(ii) Let $v \in V$ be arbitrary. Then $\mathbb{F}v + HV$ is a finite-dimensional S_H-module and therefore it is the direct sum of irreducible (and trivial) S_H-modules. Since $HV/\operatorname{ann}_{HV} H = V_H$ is S_H-irreducible, there can occur only one non-trivial summand,

$$\mathbb{F}v + HV = (\operatorname{ann} S_H) \oplus (S_H V).$$

By this construction $S_H V \subset HV$ is irreducible, hence $S_H V \cong V_H$ as S_H-modules. The above decomposition clearly holds globally, since it holds locally,

$$V = (\operatorname{ann}_V S_H) \oplus (S_H V).$$

(iii) Since S_H is spanned by elements of rank $< 4d_0$, $\dim V_H > 2^{15}(4d_0)^2$ and V_H is S_H-irreducible, Theorem 2.12(3) shows that S_H allows a standard matrix realization (as a Lie algebra of type $A_k, \ldots,$ or D_k, and V_H the corresponding tupel-space). This realization shows that S_H is spanned by elements of rank 2 (see [13, §IV.6]).

After having proved these general results on all $\mathcal{S}(Q, m)$ and $H \in \mathcal{S}(Q, m)$ we now fix H.

(iv) Set $d := \max\{\operatorname{rk} x \mid x \in H\}$. Choose $m' > \max\{8d^2, 2^{19} d_0^2\}$, fix $H_1 \in \mathcal{S}(Q, m')$ which contains $\mathcal{L}(H, 0)$. Set

$$H' := S_{H_1} + H = H_1^{(1)} + H.$$

By definition either the $\mathcal{L}(H,0)$-module $S_{H_1}V$ contains HV or $(S_{H_1}V) \cap (HV) = (0)$. In the second case $HS_{H_1}V = (0)$ holds. But then $H^{(1)}V = H^{(2)}V \subset H^{(1)}H_1^{(1)}V = (0)$ a contradiction. Consequently, $HV \subset S_{H_1}V$. In (ii) it is proved that

$$V = (\operatorname{ann}_V S_{H_1}) \oplus (S_{H_1}V).$$

Then $H(\operatorname{ann}_V S_{H_1}) \subset (\operatorname{ann}_V S_{H_1}) \cap (HV) \subset (\operatorname{ann}_V S_{H_1}) \cap (S_{H_1}V) = (0)$. As a result,

$$V = (\operatorname{ann}_V H') \oplus (H'V), \qquad H'V = S_{H_1}V.$$

Since $H'V$ is H'-irreducible (even S_{H_1}-irreducible), it follows from the above equation that

$$\operatorname{nil} H' = (0).$$

It also follows from this equation that H' acts faithfully on $H'V$. Since S_{H_1} is spanned by elements of rank 2 (see (iii)), H' is spanned by elements of rank $\leqslant \max\{d,2\}$. Proposition 2.7 now yields that H' has a unique minimal non-central ideal. Since S_{H_1} is an ideal of H' and it is simple, $S'_H := S_{H_1}$ is this minimal ideal. $\operatorname{rad} H'$ has to be central since otherwise it would contain S_{H_1}, but S_{H_1} is simple. Therefore $H' = S_{H'} \oplus C(H')$ and $S_{H'} = H'^{(1)}$.

Since $S_{H_1} = H_1^{(1)} \subset H'^{(1)}$ and $H'V$ is S_{H_1}-irreducible, it is $H'^{(1)}$-irreducible.

Recall that S_H and S_{H_1} are spanned by elements of rank 2. Then $S_{H'} = H'^{(1)} = S_{H_1} + H^{(1)} = S_{H_1} + S_H$ is spanned by elements of rank 2 as well. As in (iii) it is easily seen that $S_{H'}$ is of classical type and $H'V$ is the natural $S_{H'}$-module.

(v) As a result of all this we state the following. For every $H \in \mathcal{S}(Q,m)$ there is H' containing H such that H' satisfies

$$\text{(a), (c)–(f), and } S_{H'} \quad \text{is of type} \quad A, B, C \text{ or } D. \tag{$*$}$$

The set of all finite-dimensional subalgebras satisfying $(*)$ is therefore a local system of L. Then there is a local subsystem where all $S_{H'}$ are of the same type. Thus $\mathcal{X}$ is a local system. $\square$

We mention that the type X in Theorem 3.11 is not uniquely determined. Namely, if $X = A$ or $X = C$ then it is determined but if $X = B$ or $X = D$ then it could also be D or B (for details see [5]).

In positive characteristic p we face the problem that $\mathfrak{sl}(pk)$ has a non-trivial center, and this central extension does not split. It is also well known that in positive characteristic finite-dimensional modules over classical simple Lie algebras need not be completely reducible. In order to have a local system of algebras available, which still does have these nice properties, one has to involve more subtle constructions.

Lemma 3.12. *Let $\mathbb{F}$ be an algebraically field of characteristic $p > 3$. For every $n \in \mathbb{N}$ and every finite-dimensional subalgebra K of L there is a finite-dimensional subalgebra H containing K and fulfilling the following conditions.*

(a) $(\operatorname{nil} H)HV = H(\operatorname{nil} H)V = (0)$.
(b) *$H/\operatorname{nil} H$ has a unique non-central ideal S, and S is simple of type A_k, B_k ,C_k or D_k.*
(c) $\dim S \not\equiv 0 \mod(p)$, $\dim V_H \not\equiv 0 \mod(p)$.
(d) *V_H is S-irreducible, and is the natural module of respective dimension $k+1, 2k+1, 2k, 2k$.*

Proof. (a) Let Q, h_0, d_0 be as before, and K be any finite-dimensional subalgebra of L. We may assume that $Q \subset K$. Choose

$$m > \max\bigl\{2^{19}d_0^2,\ n+3,\ \dim(V/\operatorname{ann}_V K) + \dim(KV) + 2,\\ 2\dim(V/\operatorname{ann}_V K) + 1\bigr\}$$

and $H \in \mathcal{S}(Q, m)$ which contains K.

Since V_H is an irreducible S_H-module of dimension $> 2^{19}d_0^2$ and S_H is spanned by elements of rank $< 4d_0$, Corollary 2.11 shows that $S_H/C(H) \cap S_H$ cannot be of Cartan or Melikian type. The Classification Theorem 2.8 then yields that it is classical simple. By [7] then S_H is one of the following

$$\mathfrak{psl}(pk') \oplus C,\ \mathfrak{sl}(pk') \oplus C,\ \mathfrak{sl}(k) \oplus C \quad \text{with } p \nmid k,$$
$$B_k \oplus C,\ C_k \oplus C,\ D_k \oplus C,\ G_2 \oplus C,\ F_4 \oplus C,\ E_6 \oplus C,\ E_7 \oplus C,\ E_8 \oplus C,$$

where $C \subset C(S_H)$. Since S_H is perfect, one obtains $C = 0$. Moreover, Theorem 2.12 shows that the rank of S_H exceeds 2^{10} if S_H is a Chevalley algebra. Therefore S_H cannot be of exceptional type.

Suppose $S_H \cong \mathfrak{psl}(pk')$. Then V_H is an irreducible (but not faithful) $\mathfrak{sl}(pk')$-module, and every $y \in \mathfrak{sl}(pk')$ has V_H-rank $\leqslant 4d_0$. Therefore Theorem 2.12(3) implies that $\mathfrak{sl}(pk')$ acts faithfully, a contradiction. As a consequence, S_H is a Chevalley algebra. Similarly, Theorem 2.12(3) shows that V_H is the standard module for S_H. Thus one of the following occurs:

$$\begin{array}{lll} S_H \cong \mathfrak{sl}(k), & \dim V_H = k, & \dim S_H = (k+1)(k-1), \\ S_H \cong B_k, & \dim V_H = 2k+1, & \dim S_H = k(2k+1), \\ S_H \cong C_k, & \dim V_H = 2k, & \dim S_H = k(2k+1), \\ S_H \cong D_k, & \dim V_H = 2k, & \dim S_H = k(2k-1). \end{array}$$

We are now going to substitute H by a subalgebra which satisfies the requirements.

(b) Suppose $S_H \cong \mathfrak{sl}(k)$, where $k = \dim V_H$. Set

$$t := \dim \operatorname{ann}_{HV} K/(\operatorname{ann}_{HV} H + KV \cap \operatorname{ann}_V K)$$

and observe that $t \geqslant \dim \operatorname{ann}_{HV} K - \dim \operatorname{ann}_{HV} H - \dim KV$. One has

$$\begin{aligned}\dim V/\operatorname{ann}_V K &\geqslant \dim HV/\operatorname{ann}_{HV} K\\ &= \dim HV - \dim \operatorname{ann}_{HV} K\\ &\geqslant \dim HV - t - \dim \operatorname{ann}_{HV} H - \dim KV.\end{aligned}$$

Hence

$$t + \dim V/\operatorname{ann}_V K + \dim KV \geqslant \dim V_H.$$

By choice of m we obtain $t \geqslant 3$. Therefore there is a subspace

$$U' \subset \operatorname{ann}_{HV} K,\ U' \cap (\operatorname{ann}_{HV} H + KV) = (0), \qquad \dim U' \geqslant 3.$$

Choose a subspace $U \subset U'$ with

$$\begin{aligned}&\dim U = 0 \quad \text{if} \quad (k+1)k(k-1) \not\equiv 0 \bmod (p),\\ &\dim U = 1 \quad \text{if} \quad k+1 \equiv 0 \bmod (p),\\ &\dim U = 2 \quad \text{if} \quad k \equiv 0 \bmod (p),\\ &\dim U = 3 \quad \text{if} \quad k-1 \equiv 0 \bmod (p),\end{aligned}$$

and a subspace $W \subset HV$ satisfying

$$\operatorname{ann}_{HV} H + KV \subset W, \qquad U \cap W = (0), \qquad U + W = HV.$$

Set

$$H' := \{h \in H \mid hU = (0),\ hW \subset W\}.$$

Clearly, $K \subset H'$. Put $H'' := H'^{(1)} + K$. Denote by $\overline{}$ both the mappings $HV \to V_H$ and $H \to H/\operatorname{nil} H \subset \mathfrak{gl}(V_H)$. Then

$$V_H = \overline{U} \oplus \overline{W}, \qquad k' := \dim \overline{W} = k - \dim U.$$

By choice of U we obtain $(k'+1)k'(k'-1) \not\equiv 0 \bmod (p)$, and $k' > n$. Note that by assumption on H one has $\mathfrak{sl}(k) \cong S_H \subset H$. Therefore $\mathfrak{sl}(k) \subset \overline{H} \subset \mathfrak{gl}(k)$, hence

$$\overline{H''} \cong \mathfrak{gl}(\overline{W}) \quad \text{or} \quad \overline{H''} \cong \mathfrak{sl}(\overline{W}).$$

In any case $\overline{H''^{(1)}} \cong \mathfrak{sl}(\overline{W})$ is the unique minimal non-central ideal of $\overline{H'}$ and $\overline{W}$ is $\overline{H''^{(1)}}$-irreducible. In fact $\mathfrak{sl}(\overline{W})$ is simple as $k' \not\equiv 0 \bmod (p)$. Note that $\dim \mathfrak{sl}(\overline{W}) \not\equiv 0$, $\dim \overline{W} \not\equiv 0 \bmod (p)$. Moreover, since $\overline{W}$ is H''-irreducible, this module is annihilated by $\operatorname{nil} H''$. But then $\operatorname{nil} H''$ annihilates V_H and therefore $\operatorname{nil} H'' \subset \operatorname{ann}_{V_H} H = \operatorname{nil} H$. The respective property of $\operatorname{nil} H$ now yields

$$(\operatorname{nil} H'')H''V = H''(\operatorname{nil} H'')V = (0).$$

We note than $\overline{H''} \cong H''/\operatorname{nil} H''$. Finally we determine $V_{H''}$. Since $KV \subset W$ and $H'^{(1)}V \subset H'(HV) = H'(U+W) = H'W \subset W$, one has $H''V \subset W$. The equation $H''\overline{W} = \overline{W}$ yields

$$W \subset H''W + \operatorname{ann}_{HV} H, \qquad \operatorname{ann}_W H'' \subset \operatorname{ann}_{HV} H = \ker(\overline{}).$$

As $\operatorname{ann}_{HV} H \subset \operatorname{ann}_{HV} H''$ one obtains

$$H''V \subset W \subset H''V + \operatorname{ann}_W H'', \qquad \operatorname{ann}_W H'' = W \cap \ker(\bar{\ }),$$

and

$$V_{H''} = H''V/\operatorname{ann}_{H''V} H'' \cong W/\operatorname{ann}_W H'' = \overline{W}.$$

Thus H'' contains K and satisfies (a)–(d).

(c) Suppose $\overline{H} \cong S_H \cong \mathfrak{o}(V_H, \varphi)$ is of type B_k, C_k, D_k, and φ is a non-degenerate $\overline{H}$-invariant symmetric or skew-symmetric bilinear form on V_H. For any subspace M of V_H let $M^\perp$ denote the orthogonal space with respect to φ. We will apply the well-known equation

$$\dim V_H = \dim M + \dim M^\perp.$$

Consider $M := \operatorname{ann}_{V_H} K$. Then

$$\begin{aligned}\dim M^\perp &= \dim V_H/M = \dim V_H/\operatorname{ann}_{V_H} K \leqslant \dim HV/\operatorname{ann}_{HV} K \\ &\leqslant \dim V/\operatorname{ann}_V K.\end{aligned}$$

As we assume that $\dim V_H > 2\dim V/\operatorname{ann}_V K + 1$, this gives $\dim M/M \cap M^\perp \geqslant 2$. If $\dim \overline{H} \not\equiv 0 \mod(p)$ then set $H' := H$. Otherwise choose a 2-dimensional subspace

$$\overline{U} \subset \operatorname{ann}_{V_H} K$$

so that $\varphi|\overline{U} \times \overline{U}$ is non-degenerate. Set $\overline{W} := \overline{U}^\perp$, and

$$H' := \{h \in H \mid h\overline{U} = (0)\}.$$

Note that $V_H = \overline{U} \oplus \overline{W}$, $\varphi|\overline{W} \times \overline{W}$ is non-degenerate, $K \subset H'$. Also, $\overline{H'} \cong \mathfrak{o}(\overline{W}, \varphi|\overline{W} \times \overline{W})$ is of type $B_{k-1}, C_{k-1}, D_{k-1}$, in the respective cases. It is easy to check that $\dim \overline{H'} \not\equiv 0$, $\dim \overline{W} \not\equiv 0 \mod(p)$. As in the former case it follows that $\operatorname{nil} H' \subset \operatorname{nil} H$, hence

$$(\operatorname{nil} H')H'V = H'(\operatorname{nil} H')V = (0), \qquad \overline{H'} = H'/\operatorname{nil} H'.$$

By [7] the central extension

$$0 \to \operatorname{nil} H' \to H' \to \overline{H'} \to 0$$

splits. Thus

$$H' = S_{H'} \oplus \operatorname{nil} H',$$

where $S_{H'} \cong \overline{H'}$ is simple. Since $S_{H'}$ is perfect, one has $\overline{H'V} = \overline{H'^{(1)}V} \subset H'V_H \subset \overline{W}$, hence

$$H'V \subset W + \operatorname{ann}_{HV} H,$$

and therefore one concludes as in the former case

$$V_{H'} \cong \overline{W}.$$

Thus H' contains K and satisfies (a)–(d).

(d) Finally suppose that $S_H \cong \mathfrak{o}(V_{H,\varphi})$ is of type B_k, C_k, D_k and $\overline{H} \ncong S_H$. Every ideal J of $\overline{H}$ gives rise to an ideal of H. Since S_H is the unique minimal non-central ideal of H, this gives $\overline{S_H} \subset J$ or $J \subset \overline{C(H)}$. Since $\operatorname{Der} \overline{S_H} = \operatorname{ad} \overline{S}_H$ in the present case, one has $\overline{H} = \overline{S_H} \oplus C_{\overline{H}}(\overline{S_H})$. The preceding remark shows that

$$\overline{H} = \overline{S_H} \oplus \overline{C(H)}.$$

Due to the present assumption this gives $\overline{C(H)} \neq (0)$. Hence there is $z \in C(H) \setminus \operatorname{nil} H$, and according to Lemma 3.8 V decomposes

$$V = V^0(z) \oplus V^1(z), \quad V^1(z) \cong V_H.$$

We now proceed as in case (c). If $\dim S_H \not\equiv 0 \operatorname{mod}(p)$ then set $H' = H$. Otherwise choose a 2-dimensional subspace $\overline{U} \subset \operatorname{ann}_{V_H} K$ and $\overline{W} := U^{\perp}$. Set

$$H' := \{h \in H \mid h\overline{U} = 0,\ h\overline{W} \subset \overline{W}\}.$$

Since $\overline{C(H)} \cong \mathbb{F}\,\mathrm{Id}_{V_H}$, one has

$$\varphi(z\bar{u}, \overline{w}) = \varphi(\bar{u}, z\overline{w}) = 0 \quad \forall\, z \in \overline{C(H)},\ \bar{u} \in \overline{U},\ \overline{w} \in \overline{W}.$$

As $\overline{H} = \overline{S_H} \oplus \overline{C(H)}$, this shows that $\overline{W}$ is K-invariant, hence $K \subset H'$. Also, $\overline{H'}^{(1)} \cong \mathfrak{o}(\overline{W}, \varphi|\overline{W} \times \overline{W})$ is of type $B_{k-1}, C_{k-1}, D_{k-1}$ in the respective cases. As in the former case one concludes that

$$\dim \overline{H'}^{(1)} \not\equiv 0, \qquad \dim \overline{W} \not\equiv 0 \bmod (p),$$

$$\operatorname{nil} H' = (\operatorname{nil} H) \cap H',$$

$$H' = S_{H'} \oplus C(H'), \quad S_{H'} = \overline{H'}^{(1)}.$$

It remains to show that $V_{H'} \cong \overline{W}$. Note that

$$H'V \subset V^0(z) \cap H'V + H'V^1(z) \subset \operatorname{ann}_{H'V} H + H'HV.$$

Therefore

$$V_{H'} = H'V / \operatorname{ann}_{H'V} H' \cong H'V_H = \overline{W}.$$

Thus H' contains K and satisfies (a)–(d). $\square$

Theorem 3.13. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 3$, V an infinite-dimensional $\mathbb{F}$-vector space, and $L \subset \mathfrak{fgl}(V)$ an irreducible subalgebra. There is $X \in \{A, B, C, D\}$ such that the following is true. The family of all finite-dimensional subalgebras H of L satisfying the following conditions is a local system $\mathcal{X}^{(p)}$ of L.*

(a) $H = H^{(1)} \oplus C(H)$.
(b) $H^{(1)} \cong X_{k(H)}$ *($k(H) \geqslant 4$) is a classical simple algebra of type X.*
(c) $\dim H^{(1)} \not\equiv 0$, $\dim HV \not\equiv 0 \bmod (p)$.

(d) $\operatorname{nil} H = (0)$.
(e) $H^{(1)}$ *is spanned by elements of rank* 2.
(f) $V = (\operatorname{ann}_V H) \oplus (HV)$.
(g) HV *is* $H^{(1)}$*-irreducible, and is the natural module of respective dimension* $k(H)+1, 2k(H)+1, 2k(H), 2k(H)$ *for* $X = A, B, C, D$.

Proof. (i) Let Q, h_0, d_0 be as before and $m > 2^{19}d_0^2$. Let K be any finite-dimensional subalgebra of L which contains Q, and H an algebra associated to K and m according to Lemma 3.12. Lemma 3.9 applies to H. Thus $\operatorname{rad} H = C(H)$ and H contains a unique minimal non-central ideal S_H. Also $S_H/C(H) \cap S_H \cong S$, as S is simple of type A_k, B_k, C_k, D_k with $p \nmid k+1$ in the first case (see Lemma 3.12). By [7] the central extension splits, $S_H \cong S \oplus C(S_H)$. The perfectness of S_H yields $S_H \cong S$. All derivations of S_H are inner in the present cases, hence $H = S_H \oplus C(H)$ and therefore $S_H = H^{(1)}$.

(ii) By Lemma 3.12(d) V_H is the natural S_H-module. Thus S_H is spanned by elements of rank 2. Regarding S_H as the matrix algebra A_k, B_k, C_k, D_k, respectively, the ordinary trace form

$$\kappa(X, Y) = \operatorname{trace}(XY), \quad X, Y \in S_H,$$

is non-degenerate. Let (e_i), (e^i) be dual bases with respect to κ,

$$\kappa(e_i, e^i) = \delta_{i,j} \quad \forall\, i, j \leqslant \dim S_H.$$

Define a central element

$$z := \sum_{i=1}^{\dim S_H} e^i e_i \in C\big(U(S_H)\big).$$

Since V_H is S_H-irreducible and z commutes with S_H, one has $z|V_H = \alpha \operatorname{Id}_{V_H}$, where

$$\alpha \dim V_H = \operatorname{trace}(z|V_H) = \sum \operatorname{trace}\big(e^i e_i\big) = \dim S_H \not\equiv 0.$$

Thus Lemma 3.8 proves the splitting of V,

$$V = V^0(z) \oplus V^1(z),$$

where $S_H V^0(z) = S_H^{(1)} V^0(z) = (0)$, $V^1(z) \cong V_H$ as S_H-modules. Therefore we obtain

$$V = (\operatorname{ann}_V S_H) \oplus (S_H V), \quad S_H V \cong V_H.$$

(iii) After having proved these general results on all such algebras H we now fix such H. Set $d := \max\{\operatorname{rk} x \mid x \in H\}$. Choose $m' > \max\{8d^2, 2^{19}d_0^2\}$, fix an algebra H_1 which is associated to $\mathcal{L}(H, 0)$ and m' according to Lemma 3.12. Note that

$$Q \subset K \subset H \subset \mathcal{L}(H, 0) \subset H_1, \qquad m' > 2^{19}d_0^2.$$

Thus the results of (i), (ii) apply to H_1. Set

$$H' := S_{H_1} + H = H_1^{(1)} + H.$$

Now the arguments of (iv), (v) in the proof of Theorem 3.11 apply verbatimly. This completes the proof. $\square$

4. Classification results

Theorem 4.1. *Let $\mathbb{F}$ be any field of characteristic $\neq 2,3$, V an infinite-dimensional $\mathbb{F}$-vector space, and $L \subset \mathfrak{fgl}(V)$ an irreducible subalgebra.*

1. *L^3 is the unique minimal ideal of L.*
2. *L^3 is a simple algebra.*
3. *V is L^3-irreducible.*

Proof. (a) Let I be a non-zero subideal of L. Choose arbitrary $n \in \mathbb{N}$ and consider a local system $\mathcal{S}(Q,n)$ as described in Theorem 3.10. For any $H \in \mathcal{S}(Q,n)$, $H \cap I$ is a subideal of H. Recall that H^3 is the unique minimal non-central ideal of H, $H^3 = (H^3)^{(\infty)}$ is perfect, and $H^3/C(H) \cap H^3$ is simple.

Let J be a subideal of H such that $[H^3, J] \neq (0)$. Using induction on the length of the corresponding series $J \lhd \cdots \lhd H$ one easily proves that $H^3 \subset J$. Therefore either $[H^3, H \cap I] = (0)$ or $H^3 \subset I$. If there is $n \in \mathbb{N}$ such that

$$\mathcal{S}_{(n)} := \{H \in \mathcal{S}(Q,n) \mid H^3 \subset I\}$$

is a local system, then $L^3 \subset I$. In this case (1), (2) are established. Thus assume that no $\mathcal{S}_{(n)}$ is a local system. Then the above shows that for every $n \in \mathbb{N}$ the set

$$\mathcal{S}'_{(n)} := \{H \in \mathcal{S}(Q,n) \mid [H^3, H \cap I] = (0)\}$$

is a local system.

Take $x \in I$ arbitrary, $n > \operatorname{rk} x$ and $H \in \mathcal{S}'_{(n)}$ with $x \in H$. One has $[H^3, x] = (0)$. Since V_H is H^3-irreducible, x acts on V_H trivially or invertibly. In the second case $\operatorname{rk} x \geqslant \dim V_H > n$ holds, which contradicts the choice of n. But then $[H, x] \subset [H, \operatorname{nil} H] = (0)$. Since $\mathcal{S}'_{(n)}$ is a local system, this implies $[L, x] = 0$. But then x acts invertibly on V, a contradiction.

(b) Let U be a L^3-submodule of V. Consider an arbitrary local system $\mathcal{S}(Q,n)$ and arbitrary $H \in \mathcal{S}(Q,n)$. Since V_H is H^3-irreducible, every H^3-composition series has only one non-trivial factor. Thus H^3 acts nilpotently on V/U or U. As H^3 is perfect, it annihilates either V/U or U. Since L^3 is simple, even L^3 annihilates V/U or U. The L-irreducibility of V yields $V = L^3 V \subset U$ in the first case, and $U = (0)$ in the second. $\square$

Proposition 4.2. *Let $G \subset \mathfrak{fgl}(W)$ be not nil. For every non-nilpotent element $h \in G$ there exists a finite series of G-modules*

$$(0) = W_0 \subset U_0 \subset W_1 \subset U_1 \subset \cdots \subset W_r \subset U_r = W$$

such that

(a) *h acts nilpotently on U_i/W_i $\forall i = 0, \ldots, r$, h acts non-nilpotently on W_{i+1}/U_i $\forall i = 0, \ldots, r-1$;*
(b) *W_{i+1}/U_i is an irreducible G-module $\forall i = 0, \ldots, r-1$.*

Proof. (a) Note that the length of the series is bounded by $2r + 2 \leqslant 2\,\mathrm{rk}\,h + 2$. Therefore it is sufficient to construct U_0 and W_1, and proceed by induction. Choose n so that $\dim h^n V$ is minimal. By choice of h one has $h^n V \neq (0)$. h acts invertibly on $h^n V$.

Put U_0 the sum of all G-submodules W' of W for which $h^n W' = (0)$, set $\overline{W} := W/U_0$. Clearly $h^n U_0 = (0)$.

Let $W' \supsetneq U_0$ be a G-submodule of W. If h acts nilpotently on W'/U_0 then $h^n W' \subset U_0$, hence $h^{2n} W' = (0)$. But h acts invertibly on $h^n V$, therefore $h^n W' = h^{2n} W' = (0)$ and $W' \subset U_0$, a contradiction.

(b) Let $\mathfrak{M}$ be the set of all non-zero G-submodules of $\overline{W}$. Any ordered sequence $\overline{W}^1 \supset \overline{W}^2 \supset \cdots$ of elements of $\mathfrak{M}$ gives rise to a sequence of non-zero finite-dimensional vector spaces

$$h^n \overline{W}^1 \supset h^n \overline{W}^2 \subset \cdots$$

and therefore $\bigcap_{k \geqslant 1} \overline{W}^k \neq (0)$. By Zorn's lemma $\mathfrak{M}$ contains a minimal element $\overline{W}_1$. Set W_1 the preimage of $\overline{W}_1$ in W. $\square$

Since every finitary simple Lie algebra has a finite-dimensional non-solvable subalgebra [1] we obtain the following corollary.

Corollary 4.3. *Each finitary simple Lie algebra has a faithful finitary irreducible representation.*

Theorem 4.4. *Let $\mathbb{F}$ be an arbitrary field of characteristic $\neq 2, 3$ and $\overline{\mathbb{F}}$ its algebraic closure. Suppose V is an infinite-dimensional $\mathbb{F}$-vector space, and $L \subset \mathfrak{fgl}(V)$ is an irreducible subalgebra. Then there is a $\overline{\mathbb{F}}$-vector space $\overline{V}$ which contains V as an $\mathbb{F}$-subspace, and an irreducible $\overline{\mathbb{F}}$-subalgebra $\overline{L} \subset \mathfrak{fgl}(\overline{V})$ such that*

(1) $\overline{\mathbb{F}} V = \overline{V}$;
(2) $L \subset \overline{L} \cap \mathfrak{fgl}(V)$, $\overline{\mathbb{F}} L = \overline{L}$.

Proof. Due to Theorem 4.1 L^3 is simple. It is not nil because otherwise Theorem 3.10(b) would imply that $[L, L^3] = 0$, this contradicting the simplicity of L^3. Choose $h \in L^3$ which acts non-nilpotently on V. Note that $\overline{\mathbb{F}} \otimes_{\mathbb{F}} L$ acts finitary on $\overline{\mathbb{F}} \otimes_{\mathbb{F}} V$. Proposition 4.2 shows the existence of $\overline{\mathbb{F}} \otimes L$-modules

$$(0) = W_0 \subset U_0 \subset \cdots \subset W_r \subset U_r = \overline{\mathbb{F}} \otimes V$$

such that $1 \otimes h$ acts nilpotently on all U_i/W_i, and all W_i/U_{i-1} are $\overline{\mathbb{F}} \otimes L$-irreducible. Suppose $U_r \neq W_r$. Let

$$\pi : \overline{\mathbb{F}} \otimes V \to U_r/W_r \neq (0)$$

denote the natural $\overline{\mathbb{F}} \otimes L$-module homomorphism. Since $1 \otimes V$ is $1 \otimes L$-irreducible, and $\pi(1 \otimes V)$ generates U_r/W_r as an $\overline{\mathbb{F}}$-space, $\pi|1 \otimes V$ is injective. But $1 \otimes h$ acts non-nilpotently on $1 \otimes V$, so it acts non-nilpotently on U_r/W_r. This contradiction shows that $W_r = U_r = \overline{\mathbb{F}} \otimes V$. Set $\overline{V} := \overline{\mathbb{F}} \otimes (V/U_{r-1}) \neq (0)$, and observe that as above $1 \otimes V$ injects into $\overline{V}$. Identify V and the image of $1 \otimes V$. The mapping $\overline{\mathbb{F}} \otimes V \to \overline{V}$ gives rise to a homomorphism

$$\sigma : \overline{\mathbb{F}} \otimes L \to \mathfrak{fgl}(\overline{V}).$$

Set $\overline{L} := \sigma(\overline{\mathbb{F}} \otimes L)$. Since $1 \otimes L$ acts faithfully on $1 \otimes V$ one has that $\sigma|1 \otimes L$ is injective. Identify L with $\sigma(1 \otimes L)$. □

Having in hands Theorem 3.13, we can finish the proof of our main theorems in the same manner as in characteristic zero case. We can use either the approach of [5] or the straightforward argument from [4]. Since the ground field is algebraically closed it is reasonable to use the latter approach, which is a bit shorter.

First let us introduce some notation. Let V be a vector space over $\mathbb{F}$. A subspace Π of the dual space V^* is called *total* if

$$\operatorname{Ann}_V \Pi = \{v \in V \mid \varphi v = 0 \text{ for all } \varphi \in \Pi\} = 0.$$

The space V^* is a $\mathfrak{gl}(V)$-module under the standard action:

$$(g\varphi)v = \varphi(-gv)$$

for all $v \in V$, $\varphi \in V^*$, and $g \in \mathfrak{gl}(V)$. Let Π be a total subspace of V^*. Then

$$\mathfrak{fgl}(V, \Pi) := \bigl\{g \in \mathfrak{fgl}(V) \bigm| g\Pi \subset \Pi\bigr\}$$

is called the *finitary general linear algebra* (with respect to Π). If $\Pi = V^*$, then obviously $\mathfrak{fgl}(V, V^*) = \mathfrak{fgl}(V)$. We say that finite-dimensional subspaces $V' \subset V$ and $\Pi' \subset \Pi$ are *compatible* if $\dim V' = \dim \Pi'$ and $\operatorname{Ann}_{V'} \Pi' = 0$.

Lemma 4.5 [5, Lemma 5.7]. *Let Π be a total subspace of V^*. Then for any finite-dimensional subspaces $V_1 \subset V$ and $\Pi_1 \subset \Pi$ there exist compatible subspaces $V' \subset V$ and $\Pi' \subset \Pi$ such that $V_1 \subset V'$ and $\Pi_1 \subset \Pi'$.*

Proof. Choose any $V_2 \subset V$ compatible with Π_1. Set $V' = V_1 + V_2$. Since $\operatorname{Ann}_V \Pi = 0$, there exists a subspace $\Pi' \supseteq \Pi_1$ of Π compatible with V'. $\square$

Assume that V' and Π' are compatible, $k = \dim V' = \dim \Pi'$. Denote by $H(V', \Pi')$ the Lie algebra of all transformations g such that $gV \subset V'$ and $g\Pi \in \Pi'$. Then $H(V', \Pi')$ is a subalgebra of $\mathfrak{fgl}(V, \Pi)$ isomorphic to the finite-dimensional Lie algebra $\mathfrak{gl}(k)$. Indeed, since V' and Π' are compatible, there exist subspaces $V'' \subset V$ and $\Pi'' \subset \Pi$ such that $V = V' \oplus V''$, $\Pi = \Pi' \oplus \Pi''$ and $\Pi'' V' = \Pi' V'' = 0$. It remains to note that $g \in H(V', \Pi')$ if and only if $g\Pi'' = gV'' = 0$. We obtain the following corollary from Lemma 4.5.

Corollary 4.6. *Let Π be a total subspace of V^*. Then $\mathfrak{fgl}(V, \Pi)$ is the direct limit of its subalgebras $H(V_\tau, \Pi_\tau) \cong \mathfrak{gl}(V_\tau)$ where V_τ and Π_τ run over all compatible pairs of subspaces in V and Π. Moreover $V = V_\tau \oplus \operatorname{ann}_V H(V_\tau, \Pi_\tau)$.*

Remark 4.7. Observe that $H(V', \Pi') \subset H(V'', \Pi'')$ if and only if $V' \subset V''$ and $\Pi' \subset \Pi''$. Moreover, the corresponding embedding is *natural*, that is, one can identify these algebras with the matrix algebras $\mathfrak{gl}(k)$ and $\mathfrak{gl}(l)$ in such a way that any matrix $M \in \mathfrak{gl}(k)$ maps to $\operatorname{diag}(M, 0, \ldots, 0) \in \mathfrak{gl}(l)$.

Let Π be a total subspace of V^*. For each $g \in \mathfrak{fgl}(V, \Pi)$ one can define its trace $\operatorname{tr} g \in \mathbb{F}$ as the trace of g on the finite-dimensional subspace gV. The *finitary special linear algebra* $\mathfrak{fsl}(V, \Pi)$ is the set of all transformations $g \in \mathfrak{fgl}(V, \Pi)$ with $\operatorname{tr} g = 0$. By Corollary 4.6, $\mathfrak{fsl}(V, \Pi)$ is the direct limit of the algebras $H(V_\tau, \Pi_\tau)^{(1)} \cong \mathfrak{sl}(V_\tau)$. In particular, $\mathfrak{fsl}(V, \Pi) = \mathfrak{fgl}(V, \Pi)^{(1)}$. Recall that $\mathfrak{sl}(V_\tau)$ is simple provided $\operatorname{char} \mathbb{F} = 0$ or $\operatorname{char} \mathbb{F} = p$ and $p \nmid \dim V_\tau$. Therefore $\mathfrak{fsl}(V, \Pi)$ is simple. Since V is the direct limit of all V_τ, $\mathfrak{fsl}(V, \Pi)$ is irreducible.

When V has infinite dimension then V^* has uncountably infinite dimension; and since $\mathfrak{fsl}(V) = \mathfrak{fsl}(V, V^*)$ contains all transformations $t_{u\varphi} : v \mapsto (\varphi v)u$ with $\varphi v = 0$ $(u, v \in V, \varphi \in \Pi)$, the dimension of $\mathfrak{fsl}(V)$ is uncountably infinite. There is another finitary counterpart to the finitary special linear algebra which remains to be countably dimensional for countably dimensional V; the stable special linear algebra $\mathfrak{sl}(\infty)$. This is best introduced in terms of matrices. Every $n \times n$ matrix M can be extended to $(n+1) \times (n+1)$ matrix M' by placing M in the upper left-hand corner of M' and then bordering M with 0's. This gives us natural embeddings

$$\mathfrak{sl}(2) \to \mathfrak{sl}(3) \to \cdots \to \mathfrak{sl}(n) \to \cdots .$$

The union of these algebras is then the *stable special linear algebra* $\mathfrak{sl}(\infty)$ and is countably dimensional. Let V be a countably dimensional space with a basis $E = \{e_1, e_2, \ldots\}$ and Π be a subspace of V^* spanned by $E^* = \{e_1^*, e_2^*, \ldots\}$, the dual of the basis E. Then, clearly, $\mathfrak{fsl}(V, \Pi) \cong \mathfrak{sl}(\infty)$. One easily gets the following proposition.

Proposition 4.8 [5, Proposition 6.2]. *The algebra* $\mathfrak{fsl}(V,\Pi)$ *has countable dimension if and only if it is isomorphic to* $\mathfrak{sl}(\infty)$.

Let Φ be a non-degenerate symmetric form on V. The *finitary orthogonal algebra* $\mathfrak{fo}(V,\Psi)$ is the set of all transformations $g \in \mathfrak{fgl}(V)$ such that

$$\Phi(gv,w)+\Phi(v,gw)=0 \quad \text{for all } v,w \in V.$$

If we have a non-degenerate skew-symmetric form Ψ on V, then we obtain the *finitary symplectic algebra* $\mathfrak{fsp}(V,\Psi)$. If $\operatorname{char} F \neq 2$ and V is finite-dimensional, then $\mathfrak{fo}(V,\Phi)=\mathfrak{o}(V,\Phi)$ and $\mathfrak{fsp}(V,\Psi)=\mathfrak{sp}(V,\Psi)$ are classical simple (except some small dimensions) Lie algebras [12, Lemma 7].

Assume that $\dim V$ is infinite. Let $\{V_\tau\}$ be the set of all finite-dimensional subspaces of V with non-degenerate restrictions $\Phi_\tau = \Phi|_{V_\tau}$. For each V_τ we have $V = V_\tau \oplus V_\tau^\perp$ where $V_\tau^\perp$ is the orthogonal complement to V_τ with respect to Φ. Denote by L_τ the algebra of all elements $g \in \mathfrak{fo}(V,\Phi)$ with $gV_\tau^\perp = 0$. Clearly, $L_\tau \cong \mathfrak{o}(V_\tau,\Phi_\tau)$.

Let H be a finite-dimensional subalgebra in $\mathfrak{fo}(V,\Phi)$. Note that $\dim HV$ is finite. Since Φ is non-degenerate, there exists V_τ containing HV. Since $H \subset \mathfrak{fo}(V,\Phi)$, H leaves $V_\tau^\perp$ invariant. As $HV \subset V_\tau$, we have $HV_\tau^\perp = 0$, so $H \subset L_\tau$. Therefore $\{L_\tau\}$ is a local system of $\mathfrak{fo}(V,\Phi)$.

Note that $L_{\tau_1} \subset L_{\tau_2}$ if and only if $V_{\tau_1} \subset V_{\tau_2}$. Moreover, if $V_{\tau_1} \subset V_{\tau_2}$, then the corresponding embedding of $L_{\tau_1} \cong \mathfrak{o}(V_{\tau_1},\Phi_{\tau_1})$ into $L_{\tau_2} \cong \mathfrak{o}(V_{\tau_2},\Phi_{\tau_2})$ is natural. Clearly, $\mathfrak{fo}(V,\Phi)$ is simple. Since V is the direct limit of all V_τ, $\mathfrak{fo}(V,\Phi)$ is irreducible. Similarly we prove that $\mathfrak{fsp}(V,\Psi)$ is simple, irreducible, and is the direct limit of finite-dimensional simple symplectic subalgebras L_τ naturally embedded each into another. One also easily get the following proposition.

Proposition 4.9 [5, Proposition 6.5]. *Let V be a vector space with a non-degenerate symmetric (respectively skew-symmetric) form Φ. Then the algebra* $\mathfrak{fo}(V,\Phi)$ (*respectively* $\mathfrak{fsp}(V,\Phi)$) *is countably dimensional if and only if V is countably dimensional.*

Proof of Theorem 1.1. Let L be an infinite-dimensional finitary simple Lie algebra. By Corollary 4.3, we can assume that L is an irreducible subalgebra of $\mathfrak{fgl}(V)$ for some vector space V. Let $\mathcal{X}^{(p)}$ be the local system for L as in Theorem 3.13. Since $[L,L]=L$, we can assume that $H^{(1)}=H$ for all $H \in \mathcal{X}^{(p)}$. By Theorem 3.13, for each $H \in \mathcal{X}^{(p)}$, V can be decomposed as $V = V_H \oplus V_H^0$ where $HV_H^0 = 0$ and $V_H = HV_H$ is a natural module for $H \cong \mathfrak{sl}(V_H), \mathfrak{o}(V_H), \mathfrak{sp}(V_H)$. Consider the following cases.

(1) All $H \in \mathcal{X}^{(p)}$ have type A (special linear). Fix $H \in \mathcal{X}^{(p)}$. We have $H \cong \mathfrak{sl}(V_H)$. Let Π_H be a subspace of V^* compatible with V_H such that $\Pi_H V_H^0 = 0$. Obviously, $H = H(V_H,\Pi_H)^{(1)}$. Let $H \subset H'$. Then

$$H(V_H,\Pi_H)^{(1)} = H \subset H' = H(V_{H'},\Pi_{H'})^{(1)}.$$

Therefore $\Pi_H \subset \Pi_{H'}$. Let Π be the direct limit of all Π_H Clearly, $\mathrm{Ann}_V \Pi = 0$ and $L \subset \mathfrak{fsl}(V, \Pi)$. It remains to show that $L = \mathfrak{fsl}(V, \Pi)$. Let $g \in \mathfrak{fsl}(V, \Pi)$. By Lemma 4.5 and Remark 4.7, there is $H \in \mathcal{X}^{(p)}$ such that $gV \subset V_H$ and $g\Pi \subset \Pi_H$. Then $g \in H(V_H, \Pi_H)^{(1)} = H$, as required.

(2) All $H \in \mathcal{X}^{(p)}$ have type B or D (orthogonal). Fix $H^0 \in \mathcal{X}^{(p)}$. One can assume that $H^0 \subset H$ for all $H \in \mathcal{X}^{(p)}$. Fix $H \in \mathcal{X}^{(p)}$. Since $H \cong \mathfrak{o}(V_H)$, there exists a unique (up to a scalar multiplier) non-degenerate symmetric form Φ_H on V_H that is invariant under H. Let $H \subset H'$. Then H leaves invariant $\Phi_{H'}$ on $V_{H'}$. Therefore $\Phi_{H'}|_{V_H} = \lambda_{HH'}\Phi_H$ where $\lambda_{HH'} \in \mathbb{F}$. Note that $\lambda_{HH'} \neq 0$. For, otherwise the restriction $V_{H'}{\downarrow}H$ has at least two non-trivial composition factors isomorphic to V_H. Set

$$\overline{\Phi}_H = (1/\lambda_{H^0H})\Phi_H$$

for all H. Then $\overline{\Phi}_H|_{V_{H^0}} = \overline{\Phi}_{H^0}$ for all H. Therefore for each pair $H \subset H'$ we have $\overline{\Phi}_{H'}|_{V_H} = \overline{\Phi}_H$.

Define the form Φ on V by

$$\Phi|_{V_H} = \overline{\Phi}_H.$$

Clearly, Φ is symmetric and non-degenerate. By construction, $L \subset \mathfrak{fo}(V, \Phi)$. Let now $g \in \mathfrak{fo}(V, \Phi)$. Then $gV \subset V_H$ for some H. Since g leaves Φ invariant, $gV_H^{\perp} = 0$ where $V_H^{\perp}$ is the orthogonal complement of V_H in V. Note that we have also $HV_H^{\perp} = 0$. Since $H \cong \mathfrak{o}(V_H)$, we have $g \in H \subset L$. Therefore $L = \mathfrak{fo}(V, \Phi)$, as required.

(3) All H have type C (symplectic). Arguing as in (2), one can show that $L = \mathfrak{fsp}(V, \Psi)$, where Ψ is a non-degenerate skew-symmetric form on V. The theorem follows. □

Proof of Corollary 1.2. This follows from Propositions 4.8 and 4.9. □

Proof of Theorem 1.3. By Theorem 4.1, L^3 is simple and acts irreducible on V. It follows from Theorem 1.1 and its proof that $L^3 = \mathfrak{fsl}(V, \Pi)$, $\mathfrak{fo}(V, \Phi)$, or $\mathfrak{fsp}(V, \Psi)$. By [6, Proposition 3.2] (alternatively, see the proof of [5, Theorem 1.6]), the normalizer of L^3 in $\mathfrak{fgl}(V)$ is $\mathfrak{fgl}(V, \Pi)$, $\mathfrak{fo}(V, \Phi)$, or $\mathfrak{fsp}(V, \Psi)$, respectively. It remains to note that L normalizes L^3 and the codimension of $\mathfrak{fsl}(V, \Pi)$ in $\mathfrak{fgl}(V, \Pi)$ is 1. □

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